A NEW ALGEBRA FOR SCHOOLS

PART III

BY

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PREFACE

THIS book represents an attempt to gather up and make full use of the numerous detailed suggestions on methods of teaching Algebra, on the choice of subject-matter, on the selection of illustrations, on the construction of exercises and test-papers, etc., which have come under the author's notice since he first began (fifteen years ago) to write on the subject. It is divided into three parts. Part I deals with notation, formulae, simple equations and problems; Part II includes factors, fractions, simultaneous and quadratic equations; Part III completes the course for "additional mathematics" in School Certificate. Higher Certificate and Scholarship work is dealt with in *Advanced Algebra* (Durell and Robson).

The book, both as regards text and exercises, is written to meet the requirements of ordinary pupils. If a book includes enough (and sufficiently difficult) examples to occupy and train pupils of special ability, it must contain much that is neither required by nor is suitable for many of the others. For this reason, an appendix has been compiled, consisting of revision exercises, harder supplementary exercises and harder test-papers; and references to it have been inserted at appropriate places. This appendix will also be of use to those teachers who like to have a large range of examples from which to make their own selection for class work, and it supplies in a systematic form the material for a revision course. Both Parts I-II (bound together) and Part III may be obtained with or without the relevant portions of the appendix. Another feature to which the author attaches importance is the provision of groups of "Extra Practice" exercises at the end of Part I and of Part II for pupils who need additional "drill"; similar exercises are included in the appendix to Part III.

The initial difficulties in Algebra are mainly due to the novelty of the notation. These are best overcome by training the pupil to think in numbers when using letters and by demonstrating the practical utility of the notation by applications to formulae. These two principles have determined the selection of the subject-matter of the early chapters. Throughout the book illustrations

have been drawn from practical geometry, physics and mechanics to increase the interest in the theory and to secure variety in its problems, and this object is furthered by a free use of diagrams.

There are a few exercises in Part III (Ex. VIII. *a*, *c*, IX. *a*, *b*) which merely elaborate types of examples included in Part II. The requirements of certain examinations make it necessary to insert them, but the continuity of the course will not be broken if they are passed over.

The author acknowledges gratefully the valuable help he has received from Miss E. M. Read, Mr. G. Ayres, and Mr. A. Buckley, who have read the proofs and have made many useful suggestions. He is also indebted to Mr. R. M. Wright for some of the test-papers and for assistance with the proofs.

C. V. D.

NOTE

THE new edition contains a supplement covering the further requirements of certain examining bodies for "additional mathematics" in School Certificate and Matriculation examinations.

CONTENTS

PART III

* These details apply to the edition with Appendix. The book, as noted in the Preface, is available either with or without Appendix. Both editions are issued with Answers or without Answers.

TABLES

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	·0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
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·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

PART III

CHAPTER I

INDICES AND LOGARITHMS

Positive Integral Indices

If n is any positive integer,

a^n is a short way of writing $a \times a \times a \times \ldots$ to n factors.

Example 1. Simplify the following :

(i) $a^3 \times a^4$; (ii) $a^6 \div a^2$; (iii) $(a^2)^3$.

$$\begin{aligned}
\text{(i) } a^3 \times a^4 &= (a \times a \times a) \times (a \times a \times a \times a) \\
&= a \times a \times a \times a \times a \times a \times a = a^7.
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } a^6 \div a^2 &= \frac{a \times a \times a \times a \times a \times a}{a \times a} \\
&= a \times a \times a \times a = a^4.
\end{aligned}$$

$$\begin{aligned}
\text{(iii) } (a^2)^3 &= a^2 \times a^2 \times a^2 \\
&= (a \times a) \times (a \times a) \times (a \times a) \\
&= a \times a \times a \times a \times a \times a = a^6.
\end{aligned}$$

EXERCISE I. a

1. Obtain from first principles the simplest forms for the following :

(i) $a^2 \times a^3$; (ii) $b^5 \div b^2$; (iii) $(c^3)^2$; (iv) $d^9 \div d^3$.

2. Write down the simplest forms for the following :

(i) $a^4 \times a^2$; (ii) $b^6 \div b^3$; (iii) $x \times x^4$; (iv) $y^5 \div y$;
(v) $(c^4)^2$; (vi) $d^2 \div d^6$; (vii) $z^5 \times z^3$; (viii) $(w^5)^3$;
(ix) $a \div a^4$; (x) $(b^2)^4$; (xi) $c^8 \div c^2$; (xii) $d^5 \times d^5$.

3. What general formulae include the following special cases ?

(i) $a^3 \times a^4 = a^7$; $a^2 \times a^6 = a^8$; $a^8 \times a^3 = a^{11}$.
(ii) $a^6 \div a^2 = a^4$; $a^8 \div a^3 = a^5$; $a^9 \div a^5 = a^4$.
(iii) $(a^2)^3 = a^6$; $(a^5)^2 = a^{10}$; $(a^3)^4 = a^{12}$.

4. What is the square of x^6 ?

5. By what must y^4 be multiplied to give y^{12} ?

6. By what must z^{10} be divided to give z^5 ?

7. Simplify (i) $a^4 \times a^6 \div a^2$; (ii) $(b^2)^4 \div b^2$.

8. What is (i) the cube of $3c^3$? (ii) the square of $4d^4$?

9. Simplify (i) $(x^4)^5$; (ii) $(y^5)^4$.

10. What values of x satisfy the following equations?

(i) $x \times a^3 = a^6$; (ii) $b^8 \div x = b^2$; (iii) $b^6 \times x = b^2$.

Fractional and Negative Indices

The examples in Ex. I. a illustrate the following facts:

If m, n are positive integers,

$$\text{(i) } a^m \times a^n = a^{m+n}.$$

$$\text{(ii) } a^m \div a^n = a^{m-n}, \text{ if } m > n.$$

$$\text{(iii) } (a^m)^n = a^{mn}.$$

These results are proved by the methods used in Example 1, p. 1.

If m is not a positive integer, *e.g.* if $m = \frac{3}{4}$ or -5 or 0 or $-1\frac{1}{2}$, the symbol a^m has not yet been defined. We shall now find out what meaning must be given to it if the law

$$a^m \times a^n = a^{m+n}$$

remains true for *all* values of m and n.

Example 2. What is the meaning of $9^{\frac{1}{2}}$?

$$9^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 9^{\frac{1}{2}+\frac{1}{2}} = 9^1 = 9.$$

Since the result of multiplying $9^{\frac{1}{2}}$ by itself is 9, it follows that

$$9^{\frac{1}{2}} = \sqrt{9}.$$

Example 3. What is the meaning of $a^{\frac{1}{3}}$?

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a;$$

$$\therefore (a^{\frac{1}{3}})^3 = a.$$

Take the cube root of each side,

$$\therefore a^{\frac{1}{3}} = \sqrt[3]{a}.$$

Example 4. What is the meaning of $a^{\frac{3}{4}}$?

$$a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} = a^{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}}$$

$$= a^{\frac{3}{4} \times 4} = a^3;$$

$$\therefore (a^{\frac{3}{4}})^4 = a^3.$$

Take the fourth root of each side,

$$\therefore a^{\frac{3}{4}} = \sqrt[4]{(a^3)}.$$

We may also argue as follows :

$$a^{\frac{3}{4}} = a^{\frac{1}{4}} \times a^{\frac{1}{4}} \times a^{\frac{1}{4}} = (a^{\frac{1}{4}})^3 ;$$

but, as in Example 3, we can show that $a^{\frac{1}{4}} = \sqrt[4]{a}$.

$$\therefore\ a^{\frac{3}{4}} = (\sqrt[4]{a})^3 ;$$

$$\therefore\ a^{\frac{3}{4}} = \sqrt[4]{(a^3)} = (\sqrt[4]{a})^3.$$

For example, $16^{\frac{3}{4}} = \sqrt[4]{(16^3)} = (\sqrt[4]{16})^3$; but it is quicker to evaluate $(\sqrt[4]{16})^3$ than $\sqrt[4]{(16^3)}$, because the numbers are smaller.

Thus $\sqrt[4]{16} = 2$; $\therefore\ 16^{\frac{3}{4}} = (2)^3 = 8$.

Example 5. What is the meaning of a^{-3} ?

$$a^5 \times a^{-3} = a^{5-3} = a^2 ;$$

Divide each side by a^5, $\quad \therefore\ a^{-3} = \dfrac{a^2}{a^5}$

$$= \frac{1}{a^3}.$$

Note. We have assumed that $a \neq 0$; we cannot divide each side by a^5 if $a = 0$.

We shall not give any meaning to the symbol 0^m when m is negative or zero.

Example 6. What is the meaning of a^0 ?

$$a^3 \times a^0 = a^{3+0} = a^3 ;$$

Divide each side by a^3, $\quad \therefore\ a^0 = \dfrac{a^3}{a^3}$;

$$\therefore\ a^0 = 1.$$

Note. As in Example 5, we must suppose that $a \neq 0$.

Alternative methods may be used for Examples 5, 6 :

Assume that $a^m \div a^n$ equals a^{m-n} when $n > m$, $a \neq 0$.

Then $\quad \dfrac{a^2}{a^5} = a^{2-5} = a^{-3}$; but $\dfrac{a^2}{a^5} = \dfrac{1}{a^3}$; $\therefore\ a^{-3} = \dfrac{1}{a^3}$.

Also $\quad \dfrac{a^3}{a^3} = a^{3-3} = a^0$; but $\dfrac{a^3}{a^3} = 1$; $\therefore\ a^0 = 1$.

EXERCISE I. b

1. (i) What are the squares of x^4, x^5, x^6 ?
   (ii) What are the square roots of x^8, x^{10}, x^9 ?
2. Find x^n if
   (i) $x^n \times x^n = x^6$; (ii) $x^n \times x^n = x^5$; (iii) $x^n \times x^n = x^4$;
   (iv) $x^n \times x^n = x^3$; (v) $x^n \times x^n = x^2$; (vi) $x^n \times x^n = x$.

3. Find x^n if

(i) $x^n \times x^n \times x^n = x^6$; (ii) $x^n \times x^n \times x^n = x^3$;
(iii) $x^n \times x^n \times x^n = x$.

4. What is the cube root of (i) a^{12}; (ii) a^4; (iii) a^5?

5. Simplify (i) $b^{\frac{2}{3}} \times b^{\frac{2}{3}} \times b^{\frac{2}{3}}$; (ii) $(b^{\frac{4}{3}})^3$.
What is the meaning of $x^{\frac{2}{3}}$ and $y^{\frac{4}{3}}$?

6. Show how to find the meaning of the following:

(i) $a^{\frac{1}{4}}$; (ii) $b^{\frac{2}{3}}$; (iii) $c^{\frac{3}{2}}$; (iv) $d^{2\frac{1}{3}}$.

7. What are the squares of $a^{1\frac{1}{2}}$, $b^{2\frac{1}{2}}$, $c^{4\frac{1}{2}}$?

8. What are the cubes of $a^{1\frac{1}{3}}$, $b^{1\frac{2}{3}}$, $c^{3\frac{1}{3}}$?

9. Express as powers of x:

(i) $\sqrt{x^7}$; (ii) $\sqrt[3]{x^7}$; (iii) $\sqrt[4]{x^7}$; (iv) $\sqrt[4]{x^6}$.

10. Simplify:

(i) $16^{\frac{1}{2}}$; (ii) $8^{\frac{1}{3}}$; (iii) $8^{\frac{2}{3}}$; (iv) $9^{1\frac{1}{2}}$; (v) $81^{\frac{1}{4}}$;
(vi) $27^{\frac{2}{3}}$; (vii) $16^{1\frac{1}{4}}$; (viii) $81^{\frac{3}{4}}$; (ix) $32^{\frac{3}{5}}$; (x) $16^{1 \cdot 5}$.

11. Find x^n if

(i) $x^5 \times x^n = x^8$; (ii) $x^5 \times x^n = x^7$; (iii) $x^5 \times x^n = x^6$;
(iv) $x^5 \times x^n = x^5$; (v) $x^5 \times x^n = x^4$; (vi) $x^5 \times x^n = x^3$.

Then express as powers of x, *i.e.* in the form x^n,

(a) $\dfrac{x^3}{x^5}$; (b) $\dfrac{x^4}{x^7}$; (c) $\dfrac{x^2}{x^7}$; (d) $\dfrac{x^5}{x^5}$;

and simplify each of these fractions in the ordinary way.

12. Simplify $a^7 \times a^{-4}$; what meaning does this give for a^{-4}?

13. Simplify $b^8 \times b^{-6}$. What is the meaning of b^{-6}?

14. Simplify $c^3 \times c^{-1}$. What is the meaning of c^{-1}?

15. Simplify $x^4 \times x^0$. What is the meaning of x^0?

16. Simplify $y^0 \times y^6$. What is the meaning of y^0?

17. Simplify;

(i) $x^5 \times x^0$; (ii) $x^5 \times 1$; (iii) $x^0 \times 1$; (iv) $x^0 \times 0$.
(v) $x^5 \times 0$; (vi) $x^5 \times x$; (vii) $x^0 \times x^0$; (viii) $x^0 \times x$.

18. Simplify in two ways each of the following:

(i) $\dfrac{a^{-5} \times a^5}{a^5}$; (ii) $\dfrac{a^{-\frac{1}{2}} \times a^{\frac{1}{2}}}{a^{\frac{1}{2}}}$; (iii) $\dfrac{a^{-\frac{3}{4}} \times a^{\frac{3}{4}}}{a^{\frac{3}{4}}}$.

19. Simplify:

(i) 3^{-2}; (ii) 4^{-1}; (iii) 5^0; (iv) 2^{-3}; (v) $16^{-\frac{1}{2}}$;
(vi) $(\frac{2}{3})^{-4}$; (vii) $8^{-\frac{1}{3}}$; (viii) $(\frac{3}{5})^{-1}$; (ix) 1^{-1}; (x) $27^{-\frac{2}{3}}$.

20. Express with positive indices :

(i) a^{-4} ; (ii) $b^{-\frac{1}{2}}$; (iii) $c^{-1\frac{1}{2}}$; (iv) $\frac{1}{d^{-3}}$;

(v) $(a^3)^{-2}$; (vi) $\left(\frac{1}{b^2}\right)^{-4}$; (vii) $\left(\frac{1}{c}\right)^{-1}$; (viii) $d^{-3} \times d^{-3}$.

21. Express with root signs the following :

(i) $10^{2\frac{1}{4}}$; (ii) $10^{\cdot 5}$; (iii) $10^{\cdot 75}$; (iv) $10^{1 \cdot 25}$;

(v) $2^{\frac{3}{2}}$; (vi) $3^{1 \cdot 5}$; (vii) $5^{-\frac{1}{2}}$; (viii) $9^{-\frac{2}{3}}$.

22. If $a = 16$, $b = 9$, find the values of :

(i) $a^{\frac{1}{2}} + b^{\frac{1}{2}}$; (ii) $(a+b)^{\frac{1}{2}}$; (iii) $a^{-\frac{1}{2}} + b^{-\frac{1}{2}}$; (iv) $(a+b)^{-\frac{1}{2}}$.

23. Simplify :

(i) $9^{1 \cdot 5}$; (ii) $16^{-0 \cdot 5}$; (iii) $16^{1 \cdot 75}$; (iv) $4^{-1 \cdot 5}$;

(v) $32^{0 \cdot 4}$; (vi) $(0 \cdot 1)^{-1}$; (vii) $(\frac{1}{9})^{0 \cdot 5}$; (viii) $32^{-0 \cdot 6}$.

24. Express as a power of 9 :

(i) 81 ; (ii) 3 ; (iii) 27 ; (iv) 243 ;

(v) $\frac{1}{9}$; (vi) $\frac{1}{81}$; (vii) $\frac{1}{3}$; (viii) $\sqrt{3}$.

[*For further practice, see Appendix, Ex. F.P.* 1, *p.* 154.]

The examples in Ex. I. b show what meanings must be given to a^m when m is fractional, negative, or zero, if the law,

$$a^m \times a^n = a^{m+n},$$

holds for all values of m and n.

The results are as follows :

(i) If p, q are any positive integers,

$$\mathrm{a}^{\frac{p}{q}} = \sqrt[q]{(\mathrm{a}^p)} = (\sqrt[q]{\mathrm{a}})^p.$$

(ii) If n is any number, integral or fractional, and $a \neq 0$,

$$\mathrm{a}^{-n} = \frac{1}{\mathrm{a}^n}.$$

(iii) If $a \neq 0$, $\mathrm{a}^0 = 1.$

In particular, we have $\mathrm{a}^{\frac{1}{q}} = \sqrt[q]{\mathrm{a}}.$

The following special cases, which occur frequently, should be noted carefully :

$$a^{-1} = \frac{1}{a}; \quad a^{\frac{1}{2}} = \sqrt{a}; \quad a^{-\frac{1}{2}} = \frac{1}{\sqrt{a}}.$$

If a^m is defined as the function of a which obeys the law,

$$a^m \times a^n = a^{m+n},$$

for all values of m and n, $a \neq 0$, the results (i)-(iii) above may be proved by the methods used in Examples 4-6, p. 3.

Powers of 10

Since $10^{\frac{1}{2}} = \sqrt{10}$, we can find its value to as many places of decimals as we like by using the process for calculating square roots or by using square root tables.

Thus $10^{\frac{1}{2}} = \sqrt{10} \quad = 3{\cdot}162 \ldots .$

Similarly, $10^{\frac{1}{4}} = \sqrt{(10^{\frac{1}{2}})} = \sqrt{(3{\cdot}162)} = 1{\cdot}778 \ldots ,$

and $10^{\frac{1}{8}} = \sqrt{(10^{\frac{1}{4}})} = \sqrt{(1{\cdot}778)} = 1{\cdot}333 \ldots .$

Again $10^{\frac{3}{4}} = 10^{\frac{1}{2}} \times 10^{\frac{1}{4}} \simeq 3{\cdot}162 \times 1{\cdot}778$

$= 5{\cdot}62$, correct to 3 figures.

Or $10^{\frac{3}{4}} = \sqrt[4]{(10^3)} = \sqrt[4]{(1000)} = \sqrt{(31{\cdot}62)} \simeq 5{\cdot}623.$

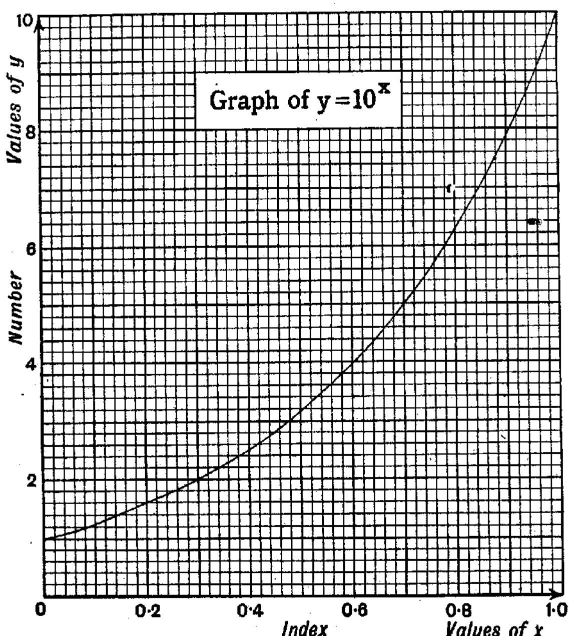


FIG. 1.

By multiplication or using square root tables, as above, it is easy to add to this list the values of $10^{\frac{3}{8}}$, $10^{\frac{5}{8}}$, $10^{\frac{7}{8}}$.

Also we have seen that $10^0 = 1$.

We can therefore construct a table of values for the function $y = 10^x$ as under :

x	0	·125	·25	·375	·5	·625	·75	·875	1
$y = 10^x$	1	1·33	1·78	2·37	3·16	4·22	5·62	7·50	10

The graph obtained by using this table is shown in Fig. 1 ; the reader should draw this graph for himself, on as large a scale as the size of the paper permits, and use it for Exercise I. c.

EXERCISE I. c

Use the graph of $y = 10^x$ to answer the following questions :

1. What are the values of :

(i) $10^{0·2}$; (ii) $10^{0·7}$; (iii) $10^{0·42}$; (iv) $10^{0·84}$?

2. Express as powers of 10 :

(i) 2 ; (ii) 3 ; (iii) 6 ; (iv) 7·8 ; (v) 1·5.

3. Find the values of a, b, c, N, in the following argument :

$$2·8 = 10^a ; \quad 3·2 = 10^b ;$$
$$\therefore \; 2·8 \times 3·2 = 10^a \times 10^b = 10^{a+b} = 10^c = N.$$

Check by multiplication.

4. Use the method of No. 3 to find approximately the value of $4·6 \times 1·6$.

5. Use the method of No. 3 to find approximately the square of 2·7.

6. Find the values of a, b, c, N in the following argument :

$$8·5 = 10^a ; \quad 3·8 = 10^b ;$$
$$\therefore \; 8·5 \div 3·8 = 10^a \div 10^b = 10^{a-b} = 10^c = N.$$

Check by division.

7. Use the method of No. 6 to find approximately the value of $7·4 \div 2·9$.

8. Find the values of a, b, N in the following arguments :

(i) $8 = 10^a$; $\therefore \; \sqrt{8} = 10^b = N$;
(ii) $6 = 10^a$; $\therefore \; \sqrt{6} = 10^b = N$;
(iii) $9 = 10^a$; $\therefore \; \sqrt[3]{9} = 10^b = N$.

Logarithms

If any number y is expressed as a power of 10, that is, if $y=10^x$, then x is called the *logarithm* of y, to *base* 10; thus, since $2 \simeq 10^{0\cdot30}$, the logarithm of 2 is 0·30, approximately; this may be written, $\log 2 \simeq 0\cdot30$.

In Ex. I. c, the logarithms of various numbers were obtained from a graph, but it was not practicable to work with more than 2 significant figures. To obtain a higher degree of accuracy, it is simpler and quicker to use tables.

Numbers between 1 and 10. The graph on p. 6 gives logarithms of numbers between 1 and 10 *only*; these logarithms lie between 0 and 1. In the same way, tables contain logarithms of numbers between 1 and 10 only; but, *to save space, decimal points are usually omitted.* The following portion of a logarithm table should be compared with the corresponding part printed in the book of Tables.

	0	1	2	...	9	1 2 3	...	7	8	9
2·9	·4624	·4639	·4654	...	·4757	1 3 4	...	10	12	13
3·0	·4771	·4786	·4800	...	·4900	1 3 4	...	10	11	13
3·1	·4914	·4928	·4942	...	·5038	1 3 4	...	10	11	12

The figures in the outer left column represent numbers between 1 and 10, those in the inner wide columns represent their logarithms. In books of Tables, the numbers 2·9, 3·0, 3·1, etc., are usually printed 29, 30, 31, etc., and their logarithms as 4624, 4771, 4914, etc., instead of ·4624, ·4771, ·4914.

The method of finding logarithms from the tables is best explained by taking examples. Thus, from the top row of the extract,

$$2\cdot90=10^{0\cdot4624}; \qquad 2\cdot91=10^{0\cdot4639}; \qquad 2\cdot92=10^{0\cdot4654}; \text{ etc.}$$

Similarly,

$$3\cdot00=10^{0\cdot4771}; \qquad 3\cdot01=10^{0\cdot4786}; \qquad 3\cdot02=10^{0\cdot4800}; \text{ etc.}$$

In this way, we can read off the logarithm of any number between 1 and 10, expressed to *three* significant figures. For numbers expressed to *four* significant figures, we could use interpolation. Thus to find the logarithm of 2·918, we have $\log 2\cdot910=0\cdot4639$ and $\log 2\cdot920=0\cdot4654$ $\therefore$ the increase in the logarithm is 0·0015.

$\therefore$ for log 2·918, the increase$\simeq\frac{8}{10}$ of ·0015 = ·0012 ;

$\therefore$ log 2·918$\simeq$0·4639 + 0·0012 = 0·4651.

To save time, the corresponding increases are shown in the narrow columns on the right, called difference columns. Thus, in the 2·9 row, in the narrow column headed 8, we find 12. This means that, if 12 is added to 4639, giving 4651, $2{\cdot}918 = 10^{0{\cdot}4651}$, the same result as was obtained above by interpolation.

Numbers of any Magnitude. In order to multiply a number by a power of 10, it is merely necessary to move the digits an appropriate number of places to the left.

$$2{\cdot}913 \times 10^4 = 29130; \quad \therefore\ 29130 = 10^{0{\cdot}4643} \times 10^4 = 10^{4+0{\cdot}4643}.$$
$$2{\cdot}913 \times 10^3 = 2913; \quad \therefore\ 2913 = 10^{0{\cdot}4643} \times 10^3 = 10^{3+0{\cdot}4643}.$$
$$2{\cdot}913 \times 10^2 = 291{\cdot}3; \quad \therefore\ 291{\cdot}3 = 10^{0{\cdot}4643} \times 10^2 = 10^{2+0{\cdot}4643}.$$
$$2{\cdot}913 \times 10^1 = 29{\cdot}13; \quad \therefore\ 29{\cdot}13 = 10^{0{\cdot}4643} \times 10^1 = 10^{1+0{\cdot}4643}.$$

In general, if a number is multiplied by 10^n, where n is any positive integer, we move the digits n places to the left. This rule may be regarded as holding also when n is a negative integer. To multiply by 10^{-2}, we move the digits (-2) places to the left, that is, 2 places to the right, since $10^{-2} = \frac{1}{10^2}$.

Thus $2{\cdot}913 \times 10^{-2} = 0{\cdot}02913$; $2{\cdot}913 \times 10^{-3} = 0{\cdot}002913$, etc.

We may therefore extend the column given above, as follows :

$$2{\cdot}913 \times 10^{-1} = 0{\cdot}2913;$$
$$\therefore\ 0{\cdot}2913 = 10^{0{\cdot}4643} \times 10^{-1} = 10^{-1+0{\cdot}4643}.$$
$$2{\cdot}913 \times 10^{-2} = 0{\cdot}02913;$$
$$\therefore\ 0{\cdot}02913 = 10^{0{\cdot}4643} \times 10^{-2} = 10^{-2+0{\cdot}4643}.$$
$$2{\cdot}913 \times 10^{-3} = 0{\cdot}002913;$$
$$\therefore\ 0{\cdot}002913 = 10^{0{\cdot}4643} \times 10^{-3} = 10^{-3+0{\cdot}4643};$$

and so on.

In this way, we can obtain the logarithms of numbers of any magnitude, large or small, although the tables contain only logarithms of numbers between 1 and 10.

The logarithm of a number less than 1 is negative, but it is always written so that the decimal portion is positive ; thus $0{\cdot}02913 = 10^{-2+0{\cdot}4643}$; we do not replace $-2 + 0{\cdot}4643$ by $-1{\cdot}5357$.

The positive decimal portion of the logarithm is called its *mantissa*, and the corresponding (positive or negative) integral portion is called its *characteristic* ; thus the mantissa of the logarithm of 0·02913 is 0·4643, and the characteristic is (-2).

We obtain the mantissa from the tables; the characteristic is found by the method of the following example.

Example 7. Express as a power of 10, (i) 753·6; (ii) 0·007536.

(i) $753\cdot6 = 7\cdot536 \times 10^2 = 10^{0\cdot8771} \times 10^2 = 10^{2\cdot8771}$.

(ii) $0\cdot007536 = 7\cdot536 \times 10^{-3} = 10^{0\cdot8771} \times 10^{-3} = 10^{-3+0\cdot8771}$.

The *same* tables can be used for the converse process.

Example 8. Find the value of

(i) $10^{0\cdot4955}$; (ii) $10^{0\cdot4958}$; (iii) $10^{1\cdot4958}$; (iv) $10^{-3+0\cdot4958}$.

(i) We have, direct from the tables, $10^{0\cdot4955} = 3\cdot13$.

(ii) The difference 3 occurs in the column headed 2,

$$\therefore\ 10^{0\cdot4958} = 3\cdot132.$$

(iii) $10^{1\cdot4958} = 10^{0\cdot4958} \times 10^1 = 3\cdot132 \times 10 = 31\cdot32$.

(iv) $10^{-3+0\cdot4958} = 10^{0\cdot4958} \times 10^{-3} = 3\cdot132 \times 10^{-3} = 0\cdot003132$.

Example 9. Find the value of $10^{2\cdot1561}$.

From the tables, $10^{0\cdot1553} = 1\cdot43$. We now have to allow for a difference of 8. Since 8 itself does not appear in the difference column, we take the nearest number which does so, in this case 9, which is in the column headed 3.

$$\therefore\ 10^{0\cdot1561} = 1\cdot433;\quad \therefore\ 10^{2\cdot1561} = 1\cdot433 \times 10^2 \doteq 143\cdot3.$$

Example 10. Find the value of $10^{1\cdot8367}$.

From the tables, $10^{0\cdot8363} = 6\cdot86$. We now have to allow for a difference of 4; since 4 appears in the difference column both under 6 and under 7, we may take either 6 or 7. [Either 5-figure tables must be used or further calculations must be made to discover which gives the closer approximation; and it is not worth while doing this because *if 4-figure tables are being used, the fourth figure in the answer is not necessarily correct.*]

Thus $10^{0\cdot8367} \simeq 6\cdot866$, $\therefore\ 10^{1\cdot8367} \simeq 68\cdot66$.

EXERCISE I. d

[*Four-figure tables should be used in this and the following exercises.*]

Express as powers of 10 the following numbers:

1. (i) 7·4; (ii) 7·5; (iii) 7·47; (iv) 7·48; (v) 7·476.

2. (i) 5; (ii) 5·04; (iii) 5·05; (iv) 5·045; (v) 5·049.

3. (i) 2·68; (ii) 8·04; (iii) 5·59; (iv) 9·01.

4. (i) 7·683; (ii) 4·015; (iii) 8·706; (iv) 5·008.

5. (i) 60 ; (ii) 70 ; (iii) 400 ; (iv) 9000 ; (v) 0·08.

6. (i) 53 ; (ii) 49 ; (iii) 7200 ; (iv) 0·0064 ; (v) 0·26.

7. (i) 427 ; (ii) 8340 ; (iii) 0·0672 ; (iv) 0·518.

8. (i) 59·3 ; (ii) 90·7 ; (iii) 0·00105 ; (iv) 0·702.

9. (i) 5134 ; (ii) 125·4 ; (iii) 0·07142 ; (iv) 0·0003182.

10. (i) 10·07 ; (ii) 3002 ; (iii) 0·8053 ; (iv) 0·004007.

What are the numbers whose logarithms are as follows :

11. (i) 0·9031 ; (ii) 0·5563 ; (iii) 0·0792 ; (iv) 0·2041.

12. (i) 0·6767 ; (ii) 0·7825 ; (iii) 0·1004 ; (iv) 0·943.

13. (i) 0·5408 ; (ii) 0·7607 ; (iii) 0·3173 ; (iv) 0·6029.

14. (i) 1·8633 ; (ii) 3·6628 ; (iii) $-2+0·3802$;
(iv) $-1+0·9395$.

15. (i) 2·8235 ; (ii) 4·3766 ; (iii) $-1+0·8669$;
(iv) $-3+0·0864$.

16. (i) 1·8451 ; (ii) 3·0043 ; (iii) $-2+0·3075$;
(iv) $-4+0·847$.

17. (i) 2·6603 ; (ii) 5·9762 ; (iii) $-3+0·6007$;
(iv) $-1+0·1473$.

18. (i) 5·6027 ; (ii) 2·4765 ; (iii) $-2+0·0033$;
(iv) $-5+0·9035$.

Positive Indices

Example 11. Find the value of $46·87 \times 3·056$.

$$\begin{aligned} 46·87 \times 3·056 &= 10^{1·6709} \times 10^{0·4852} \\ &= 10^{1·6709+0·4852} \\ &= 10^{2·1561} \\ &= 1·433 \times 10^2 = \mathbf{143·3}. \end{aligned}$$

$$\begin{array}{r} 1·6709 \\ \underline{0·4852} \\ \underline{2·1561} \end{array}$$

Rough estimate : $50 \times 3 = 150$.

Example 12. Find the value of $\dfrac{5728}{83·42}$.

$$\begin{aligned} 5728 \div 83·42 &= 10^{3·7580} \div 10^{1·9213} \\ &= 10^{3·7580-1·9213} \\ &= 10^{1·8367} \\ &= 6·866 \times 10^1 = \mathbf{68·66}. \end{aligned}$$

$$\begin{array}{r} 3·7580 \\ \underline{1·9213} \\ \underline{1·8367} \end{array}$$

Rough estimate : $\frac{6000}{80} = 75$.

Example 13. Find the value of $\dfrac{49080 \times 24\cdot2}{270\cdot4 \times 3\cdot135}$.

$$\text{The expression} = \frac{10^{4\cdot6909} \times 10^{1\cdot3838}}{10^{2\cdot4320} \times 10^{0\cdot4962}}$$
$$= \frac{10^{4\cdot6909+1\cdot3838}}{10^{2\cdot4320+0\cdot4962}}$$
$$= \frac{10^{6\cdot0747}}{10^{2\cdot9282}} = 10^{6\cdot0747-2\cdot9282}$$
$$= 10^{3\cdot1465}$$
$$= 1\cdot401 \times 10^{3} = \mathbf{1401}.$$

$$\begin{array}{r|r} 4\cdot6909 & 2\cdot4320 \\ \underline{1\cdot3838} & \underline{0\cdot4962} \\ 6\cdot0747 & \underline{2\cdot9282} \\ \underline{2\cdot9282} & \\ \underline{3\cdot1465} & \end{array}$$

Rough estimate: $\dfrac{50000 \times 20}{300 \times 3} = \dfrac{10^6}{900} \simeq 1000.$

Example 14. Find the value of $(19\cdot07)^4$.

$$(19\cdot07)^4 = (10^{1\cdot2804})^4 = 10^{1\cdot2804\times4}$$
$$= 10^{5\cdot1216}$$
$$= 1\cdot323 \times 10^5 = \mathbf{132300}.$$

$$\begin{array}{r} 1\cdot2804 \\ \underline{4} \\ \underline{5\cdot1216} \end{array}$$

Rough estimate : $20^4 = 160000$.

Example 15. Find the value of $\sqrt[3]{(873\cdot5)}$.

$$\sqrt[3]{(873\cdot5)} = (10^{2\cdot9412})^{\frac{1}{3}}$$
$$= 10^{2\cdot9412\times\frac{1}{3}} = 10^{\cdot9804}$$
$$= \mathbf{9\cdot558}.$$

$$\begin{array}{r|l} 3 & \underline{2\cdot9412} \\ & \underline{\cdot9804} \end{array}$$

Rough check : $10^3 = 1000$, $9^3 = 729$.

Until the argument is thoroughly understood, the method of writing each number as a power of 10, which has been used in the above examples, should be followed, any necessary subsidiary calculations (addition, subtraction, etc.) being carried out neatly in side columns as shown. If the expressions are complicated, as in Example 13, it is clearer to use two side columns, one for logarithms of the numerator, the other for logarithms of the denominator.

We shall now repeat two of these examples to show how the work should be arranged, *after* the principles have been grasped.

Example 11.* Find the value of $46\cdot87 \times 3\cdot056$.

Number	Logarithm
46·87	1·6709
3·056	0·4852
Product	2·1561

$\therefore$ Product $= 1\cdot433 \times 10^2 = \mathbf{143\cdot3}$.

Example 13.* Find the value of $\dfrac{49080 \times 24{\cdot}2}{270{\cdot}4 \times 3{\cdot}135}$.

Number	Logarithm
49080	4·6909
24·2	1·3838
Numerator	6·0747
Denominator	2·9282
Expression	3·1465

Number	Logarithm
270·4	2·4320
3·135	0·4962
Denominator	2·9282

$\therefore$ expression $= 1{\cdot}401 \times 10^3 = \mathbf{1401}$.

EXERCISE I. e

Find the values of the following :

1. $2{\cdot}284 \times 17{\cdot}93$.
2. $5862 \times 11{\cdot}07$.
3. $64{\cdot}72 \times 1{\cdot}083$.
4. $85{\cdot}32 \div 6{\cdot}907$.
5. $173 \div 86{\cdot}49$.
6. $7859 \div 9{\cdot}073$.
7. $13300 \times 7{\cdot}59$.
8. $(20{\cdot}68)^2$.
9. $70400 \div 86{\cdot}04$.
10. $(409{\cdot}3)^3$.
11. $100 \div 3{\cdot}726$.
12. $(5{\cdot}08)^5$.
13. $1000 \div 89{\cdot}07$.
14. $(10{\cdot}02)^4$.
15. $11000 \div 8080$.
16. $8{\cdot}637 \times 18{\cdot}07 \times 108{\cdot}7$.
17. $13{\cdot}54 \times 66{\cdot}08 \div 95{\cdot}39$.
18. $\dfrac{14{\cdot}67 \times 112}{883{\cdot}6}$.
19. $\dfrac{8709}{3{\cdot}142 \times 75{\cdot}62}$.
20. $48{\cdot}5 \times (73{\cdot}2)^2 \times 1200$.
21. $(6{\cdot}734)^3 \times 8{\cdot}45 \div 27{\cdot}39$.
22. $\dfrac{8{\cdot}624 \times 729}{30{\cdot}72 \times 9{\cdot}134}$.
23. $\dfrac{(15{\cdot}09)^2}{8{\cdot}086 \times 12{\cdot}07}$.
24. $(7700)^3 \div (909)^2$.
25. $16{\cdot}14 \times (70{\cdot}33)^3 \div 19000$.
26. $\dfrac{7{\cdot}314 \times 20000}{(50{\cdot}08)^2}$.
27. $\dfrac{107{\cdot}7 \div 8{\cdot}35}{2{\cdot}47 \times 3{\cdot}064}$.
28. $\frac{4}{3} \times 3{\cdot}142 \times (2{\cdot}105)^3$.
29. $2\frac{3}{4} \times 191 \div (1\frac{2}{3} \times 11{\cdot}3)$.
30. $100 \div (2{\cdot}007)^4$.
31. $(3{\cdot}162 \times 6{\cdot}928 \div 4{\cdot}899)^2$.
32. (i) $\sqrt{(3{\cdot}724)}$; (ii) $\sqrt{(37{\cdot}24)}$; (iii) $\sqrt{(372{\cdot}4)}$; (iv) $\sqrt{(3724)}$.
33. (i) $\sqrt[3]{(7{\cdot}428)}$; (ii) $\sqrt[3]{(74{\cdot}28)}$; (iii) $\sqrt[3]{(742{\cdot}8)}$; (iv) $\sqrt[3]{(7428)}$.
34. $\sqrt{10}$.
35. $\sqrt[3]{9}$.
36. $\sqrt{20}$.
37. $\sqrt[4]{50}$.
38. $12{\cdot}07 \times \sqrt{(18{\cdot}14 \times 29)}$.
39. $\sqrt{(3{\cdot}724)^3} \div \sqrt[3]{100}$.
40. $\sqrt{\left(\dfrac{86{\cdot}47}{9{\cdot}173}\right)} \div 2{\cdot}75$.
41. $\dfrac{37 \times \sqrt{(117)}}{(4{\cdot}312)^3}$.

42. $\sqrt{\left(\frac{1700 \times 36{\cdot}52}{41 \times 10{\cdot}07}\right)}$.

43. $\sqrt[3]{\left(\frac{8{\cdot}626 \times 110}{10{\cdot}1 \times 30{\cdot}03}\right)}$.

44. $2\sqrt[3]{(100)} \div \sqrt{10}$.

45. $\sqrt{(3\frac{1}{4} \times 77{\cdot}8 \times 13\frac{2}{11})}$.

[*For further practice, see Appendix, Ex. F.P. 2, p.* 165.]

EXERCISE I. f

[*Throughout this Exercise, take* $\pi = 3{\cdot}1416 = 10^{\cdot 4971}$.]

1. The area of a circle of radius r in. is A sq. in. where $A = \pi r^2$ and $r = \sqrt{\left(\frac{A}{\pi}\right)}$.

(i) Find A if $r = 3{\cdot}07$; (ii) find r if $A = 10{\cdot}5$.

2. Neglecting air-resistance, a stone falls s feet in t seconds where $s = \frac{1}{2}gt^2$ and $t = \sqrt{\left(\frac{2s}{g}\right)}$; $g = 32{\cdot}2$.

(i) Find s if $t = 1{\cdot}35$; (ii) find t if $s = 19{\cdot}8$.

3. The area of the surface of a sphere of radius r in. is A sq. in. where $A = 4\pi r^2$.

(i) Find A if $r = 2{\cdot}073$; (ii) find r if $A = 1000$.

4. The volume of a sphere of radius r in. is V cu. in. where $V = \frac{4}{3}\pi r^3$ and $r = \sqrt[3]{\left(\frac{3V}{4\pi}\right)}$.

(i) Find V when $r = 13{\cdot}26$; (ii) find r when $V = 138$.

5. The time of a complete oscillation of a pendulum of length l feet is t sec. where $t = 2\pi\sqrt{\left(\frac{l}{g}\right)}$ and $l = \frac{gt^2}{4\pi^2}$; $g = 32{\cdot}2$.

(i) Find t if $l = 100$; (ii) find l if $t = 2\frac{1}{4}$.

6. The volume of a circular cone, base diameter d in., height h in., is V cu. in. where $V = \frac{\pi d^2 h}{12}$.

(i) Find V if $d = 3\frac{1}{4}$, $h = 4\frac{3}{4}$; (ii) find d if $V = 100$, $h = 7{\cdot}33$.

7. Find s from the formula, $s = \frac{Fgt^2}{2W}$, if $F = 425$, $g = 32{\cdot}2$, $t = 17$, $W = 740$.

8. At a height of h feet, the distance of the horizon is d miles where $d = 1{\cdot}42\sqrt{h}$.

(i) Find d when $h = 360$; (ii) find h when $d = 10$.

9. The weight of a brass cylinder, height h cm., base diameter d cm., is w gm. where $w = 2{\cdot}09\pi d^2 h$.

(i) Find w if $h = 6{\cdot}35$, $d = 5{\cdot}84$; (ii) find h if $w = 217$, $d = 3{\cdot}26$; (iii) find d if $w = 1000$, $h = 7\frac{1}{4}$.

10. The current which will fuse a certain wire of diameter d mm. is C amperes, where $C = \sqrt{(160d^3)}$; find C if $d = 2 \cdot 30$.

11. Find k from the formula, $k = \frac{W}{(D+d)(D-d)}$, if $W = 23 \cdot 4$ when $D = 4\frac{1}{4}$ and $d = 0 \cdot 1$.

12. Find correct to one significant figure, (i) 5^{21}; (ii) 3^{30}.

13. The speed of a paddle steamer, of cross-section S sq. ft., when the engines work at P horse-power, is v knots, where $v = \sqrt[3]{\left(\frac{620P}{S}\right)}$.

Find v if $P = 1560$ and $S = 648$.

14. The volume of a metal tube, l in. long, outer radius r in., thickness t in., is V cu. in. where $V = \pi lt(2r - t)$.

(i) Find V if $l = 7 \cdot 05$, $r = 8$, $t = 1 \cdot 3$;

(ii) find l if $V = 120$, $r = 5 \cdot 4$, $t = 1 \cdot 15$.

15. The amount of £P at r per cent. per annum compound interest is £A, where $A = P\left(1 + \frac{r}{100}\right)^n$.

Find A to the nearest integer, if $P = 185$, $r = 3\frac{1}{2}$, $n = 12$.

16. The volume of a segment of a sphere of radius r in., height h in. is V cu. in., where $V = \pi h^2(r - \frac{1}{3}h)$. Find V if $r = 5 \cdot 42$, $h = 2 \cdot 85$.

17. The surface S sq. in. and the volume V cu. in. of a sphere are connected by the relation, $S^3 = 36\pi V^2$.

(i) Find S when $V = 37$; (ii) find V when $S = 19$.

18. If the area of the total surface of a circular cone is S sq. in., its maximum volume, V cu. in., is given by the relation

$$V = \frac{1}{3}\sqrt{\left(\frac{S^3}{8\pi}\right)}.$$

(i) Find V if $S = 15$; (ii) find S if $V = 3 \cdot 2$.

19. If $\frac{4}{3}\pi kr^3 = 100$ and if $k = 7 \cdot 7$, find the value of $4\pi r^2$.

20. The area of an equilateral triangle, side x in. long, is A sq. in., where $A = \frac{1}{4}x^2\sqrt{3}$.

(i) Find A when $x = 2 \cdot 7$; (ii) find x when $A = 10$.

Negative Indices. If a number between 0 and 1 is expressed as a power of 10, the index is negative, see p. 9, but is always written so that the decimal part is positive.

Thus $0 \cdot 00874 = 8 \cdot 74 \times 10^{-3} = 10^{0 \cdot 9415} \times 10^{-3} = 10^{-3+0 \cdot 9415}$.

For brevity, the index $-3 + 0 \cdot 9415$ is written $\bar{3} \cdot 9415$ (pronounced bar 3, point 9415), the " minus " being placed above the

3 to show that it refers only to the 3, and not to ·9415. But, at first, the reader will find it easier to write out such numbers in full.

Example 16. Find the value of $\frac{378{\cdot}4}{0{\cdot}00624}$.

Number	Logarithm
378·4	2·5780
0·00624	$\bar{3}{\cdot}7952$
Expression	4·7828

$$\begin{array}{r} 2+{\cdot}5780 \\ -3+{\cdot}7952 \\ \hline 4+{\cdot}7828 \end{array}$$

$\therefore$ expression $=60640$.

Some practice is needed in subtraction with such numbers as occur here. For the decimal portion, use your ordinary method: it may be either of those given below.

If there is any difficulty with the **integers**, use the rule, change the sign of the lower line and add.

Equal Additions | *Decomposition*

$$\begin{array}{r} 2+1{\cdot}5780 \\ (-3+1)+0{\cdot}7952 \\ \hline 4+0{\cdot}7828 \end{array} \qquad \begin{array}{r} (2-1)+1{\cdot}5780 \\ -3+0{\cdot}7952 \\ \hline 4+0{\cdot}7828 \end{array}$$

It is not of course necessary to set out the work in such detail as shown here; the object of these examples is to show what argument must be employed.

Example 17. Express with the decimal part positive,

(i) $\bar{2}{\cdot}8+\bar{3}{\cdot}7$; (ii) $0-1{\cdot}32$; (iii) $0-\bar{3}{\cdot}41$;

(iv) $\bar{2}{\cdot}7\times 3$; (v) $\bar{6}{\cdot}48\div 3$; (vi) $\bar{4}{\cdot}82\div 3$.

(i) $(-2+0{\cdot}8)+(-3+0{\cdot}7)=(-5+1{\cdot}5)=(-4+0{\cdot}5)=\bar{4}{\cdot}5$.

(ii) $\begin{array}{r} 0+0{\cdot}00 \\ 1+0{\cdot}32 \\ \hline (-2)+0{\cdot}68 \end{array} =\bar{2}{\cdot}68.$ (iii) $\begin{array}{r} 0+0{\cdot}00 \\ (-3)+0{\cdot}41 \\ \hline 2+0{\cdot}59 \end{array} =2{\cdot}59$

(iv) $(-2+0{\cdot}7)\times 3=-6+2{\cdot}1=-4+0{\cdot}1=\bar{4}{\cdot}1$.

(v) $(-6+0{\cdot}48)\div 3=-2+0{\cdot}16=\bar{2}{\cdot}16$.

(vi) $(-4+0{\cdot}82)\div 3=(-6+2{\cdot}82)\div 3$

$=-2+0{\cdot}94=\bar{2}{\cdot}94$.

Example 18. Find the value of $(0{\cdot}07681)^3$.

Number	Logarithm	
$(0{\cdot}07681)^3$	$\bar{2}{\cdot}8855 \times 3$	$\bar{4}{\cdot}6565$

$$\begin{array}{r} -2+0{\cdot}8855 \\ 3 \\ \hline -6+2{\cdot}6565 \end{array}$$

$\therefore$ expression $= 4{\cdot}534 \times 10^{-4} = \mathbf{0{\cdot}0004534}$.

Example 19. Find the value of $\sqrt[4]{(0{\cdot}8635)}$.

Number	Logarithm	
$\sqrt[4]{(0{\cdot}8635)}$	$\bar{1}{\cdot}9363 \div 4$	$\bar{1}{\cdot}9841$

$$\begin{array}{r} -1+0{\cdot}9363 \\ 4\,|\;\underline{-4+3{\cdot}9363} \\ -1+0{\cdot}9841 \end{array}$$

$\therefore$ expression $= 9{\cdot}640 \times 10^{-1} = \mathbf{0{\cdot}9640}$.

Example 20. Find the value of $\dfrac{(0{\cdot}07241)^4 \times \sqrt{(0{\cdot}6273)}}{4\frac{3}{5} \times (0{\cdot}0726)^3 \times \sqrt[3]{(0{\cdot}8324)}}$.

Number	Logarithm	
$(0{\cdot}07241)^4$	$\bar{2}{\cdot}8598 \times 4$	$\bar{5}{\cdot}4392$
$\sqrt{(0{\cdot}6273)}$	$\bar{1}{\cdot}7975 \div 2$	$\bar{1}{\cdot}8987$
Numerator		$\bar{5}{\cdot}3379$
$4{\cdot}6$		$\cdot6628$
$(0{\cdot}0726)^3$	$\bar{2}{\cdot}8609 \times 3$	$\bar{4}{\cdot}5827$
$\sqrt[3]{(0{\cdot}8324)}$	$\bar{1}{\cdot}9203 \div 3$	$\bar{1}{\cdot}9734$
Denominator		$\bar{3}{\cdot}2189$
Expression		$\bar{2}{\cdot}1190$

$$\begin{array}{r} \bar{5}{\cdot}3379 \\ \bar{3}{\cdot}2189 \\ \hline \bar{2}{\cdot}1190 \end{array}$$

$\therefore$ expression $= 1{\cdot}315 \times 10^{-2} = \mathbf{0{\cdot}01315}$.

Example 21. Find the value of $\dfrac{1}{\sqrt[3]{(0{\cdot}0479)}}$.

Number	Logarithm	
1		$0{\cdot}0000$
$\sqrt[3]{(0{\cdot}0479)}$	$\bar{2}{\cdot}6803 \div 3$	$\bar{1}{\cdot}5601$
Expression		$0{\cdot}4399$

$\therefore$ expression $= \mathbf{2{\cdot}754}$.

Example 22. Find the value of $(0\cdot0864)^{-1\cdot4}$.

Number	Logarithm	
$(0\cdot0864)^{-1\cdot4}$	$\bar{2}\cdot9365\times(-1\cdot4)$	$1\cdot4889$

$(-2+0\cdot9365)\times(-1\cdot4)$
$=2\cdot8-1\cdot3111$
$=1\cdot4889$

$$\begin{array}{r} 0\cdot9365 \\ 1\cdot4 \\ \hline \cdot9365 \\ \cdot3746 \\ \hline 1\cdot3111 \end{array}$$

$\therefore$ expression $=3\cdot082\times10=30\cdot82$.

EXERCISE I. g

Express the following so that the decimal part is positive:

1. $3\cdot4+\bar{5}\cdot2$.
2. $4\cdot7+\bar{2}\cdot5$.
3. $\bar{5}\cdot6+2\cdot7$.
4. $\bar{2}\cdot8+\bar{3}\cdot5$.
5. $\bar{3}+1\cdot2$.
6. $6\cdot7-\bar{2}\cdot4$.
7. $\bar{3}\cdot8-1\cdot5$.
8. $4\cdot3-\bar{1}\cdot5$.
9. $3\cdot7-5\cdot2$.
10. $2\cdot5-6\cdot8$.
11. $\bar{4}\cdot7-\bar{1}\cdot3$.
12. $\bar{4}\cdot3-\bar{2}\cdot7$.
13. $0-2\cdot4$.
14. $\bar{4}\cdot3-2\cdot7$.
15. $\bar{2}\cdot6-4\cdot8$.
16. $0-\bar{3}\cdot6$.
17. $\bar{2}\cdot7\times3$.
18. $\bar{1}\cdot8\times4$.
19. $\bar{3}\cdot4\times(-2)$.
20. $\bar{3}\cdot8\times(-3)$.
21. $\bar{2}\cdot8\times2\cdot3$.
22. $\bar{6}\cdot8\div2$.
23. $\bar{3}\cdot6\div2$.
24. $\bar{2}\cdot8\div3$.
25. $\bar{6}\cdot4\div4$.
26. $\bar{2}\div5$.
27. $\bar{4}\cdot7\div3$.
28. $1\cdot4\div(-2)$.
29. $\bar{2}\cdot5\div(-3)$.
30. $\bar{9}\cdot6\div(-7)$.

EXERCISE I. h

Find the values of the following:

1. $0\cdot639\times837$.
2. $0\cdot746\times0\cdot646$.
3. $0\cdot0805\times0\cdot707$.
4. $0\cdot815\div53\cdot6$.
5. $0\cdot574\div0\cdot0231$.
6. $0\cdot048\div0\cdot723$.
7. $347\div0\cdot575$.
8. $1\div0\cdot0274$.
9. $46\div0\cdot0067$.
10. $(0\cdot4726)^2$.
11. $(0\cdot0813)^3$.
12. $(0\cdot1071)^4$.
13. $\sqrt{(0\cdot0834)}$.
14. $\sqrt{(0\cdot0076)}$.
15. $\sqrt[3]{(0\cdot6438)}$.
16. $\sqrt[3]{(0\cdot0405)}$.
17. $\sqrt[5]{(0\cdot0082)}$.
18. $\sqrt[3]{(0\cdot0947)^2}$.
19. $\dfrac{1}{387\cdot2}$.
20. $\dfrac{1}{\sqrt{(0\cdot828)}}$.
21. $\dfrac{1}{(0\cdot271)^3}$.
22. $\dfrac{1}{\sqrt[3]{10}}$.
23. $(84\cdot35)^{-\frac{1}{2}}$.
24. $(0\cdot607)^{-\frac{1}{3}}$.
25. $\dfrac{7835}{6942\times3781}$.
26. $\dfrac{4\cdot762\times\cdot003791}{50\cdot09\times110\cdot7}$.
27. $\dfrac{0\cdot8714\times0\cdot00652}{1837\times0\cdot0797}$.
28. $\dfrac{1}{1630\times0\cdot073\times0\cdot8624}$.

29. $\dfrac{4\cdot876 \times (0\cdot07304)^2}{(0\cdot5162)^3}$.

30. $\dfrac{0\cdot3916 \times \sqrt{(0\cdot8654)}}{(17\cdot04)^2 \times 0\cdot108}$.

31. $\sqrt{\left(\dfrac{8\cdot63 \times 0\cdot00627}{0\cdot1924 \times 3\cdot416}\right)}$.

32. $\sqrt{\left(\dfrac{0\cdot4717 \times 0\cdot0823}{29\cdot44 \times 0\cdot1}\right)}$.

33. $\dfrac{\sqrt{(0\cdot7624)} \times \sqrt[3]{(0\cdot3971)}}{(0\cdot878)^3 \times (1\cdot01)^2}$.

34. $\dfrac{1}{(0\cdot0263)^3 \times \sqrt[3]{(0\cdot0472)}}$.

35. $(6\cdot73)^{-0\cdot4}$.
36. $(0\cdot815)^{-1\cdot5}$.
37. $(651\cdot3)^{-2\cdot4}$.

38. $(0\cdot007)^{-\frac{2}{3}}$.
39. $(0\cdot05)^{3\cdot3}$.
40. $(0\cdot1)^{0\cdot1}$.

[*For further practice, see Appendix, Ex. F.P. 3, p. 156.*]

EXERCISE I. j

[*Throughout this Exercise, take log* $\pi = 0\cdot4971$.]

1. The volume of metal in a tube h in. long, external radius R in., internal radius r in., is $\pi h(\mathrm{R}+r)(\mathrm{R}-r)$ cu. in.; find the volume if $h=7\cdot524$, $\mathrm{R}=6\cdot173$, $r=5\cdot948$.

2. Find the value of $\sqrt{\{(0\cdot867)^2-(0\cdot659)^2\}}$.

3. Find the value of $2\pi\sqrt{\left(\dfrac{l}{g}\right)}$,

(i) if $l=2\cdot57$, $g=32\cdot2$; (ii) if $l=78\cdot3$, $g=981$.

4. Find r if $\pi r^2=1$. Interpret the result.

5. For wrought iron shafting, the diameter of the shaft d in., the horse-power transmitted H, and the number of revolutions per minute n are connected by the relation $d=4\cdot02\sqrt[3]{\left(\dfrac{\mathrm{H}}{n}\right)}$; find d if $\mathrm{H}=63$, $n=175$.

6. Find the value of $\sqrt{\left(\dfrac{1\cdot27\mathrm{M}}{hx}\right)}$ if $\mathrm{M}=2\cdot51$, $h=15$, $x=8\cdot9$.

7. Find the value of $\sqrt{\left\{\dfrac{(s-b)(s-c)}{s(s-a)}\right\}}$ if $a=3\cdot15$, $b=4\cdot26$, $c=5\cdot63$ and $s=\frac{1}{2}(a+b+c)$.

8. Find r if $\frac{4}{3}\pi r^3=1$. Interpret the result.

9. If $x^3=y^3+z^3$, find x when $y=0\cdot827$, $z=0\cdot619$.

10. Find the value of $\dfrac{1}{55}\left(\dfrac{76\cdot2}{304\cdot3}+\dfrac{4\cdot5}{5\cdot69}\right)$.

11. The velocity of sound in air at temperature $t°$ C., when the pressure of the air is p lb. per sq. in. and the standard density is d lb. per cu. ft., is v feet per second, where

$$v=\sqrt{\left\{45\cdot08 \times \frac{144p}{d} \times \left(1+\frac{t}{273}\right)\right\}}.$$

Find v if $p=14\cdot7$, $d=0\cdot0809$, $t=16$.

12. Find the value of $\frac{3300t}{d}\left(5-\frac{l+12}{60t}\right)$ if $t=\frac{1}{8}$, $d=4\cdot75$, $l=2\cdot8$.

13. Find the value of $5\cdot25k^3d^{-4}$ when $k=2\cdot085$, $d=0\cdot2173$.

14. The safe width of a dam at a depth of x feet below water level is $\sqrt{\left\{\frac{0\cdot05x^3}{9+0\cdot03x}\right\}}$ feet. How wide should it be at a depth of $13\frac{1}{2}$ feet ?

15. The expression $\frac{x}{(1+x^2)^{2\cdot5}}$ is greatest when $x=0\cdot5$; what is its greatest value ?

16. The bore of a pipe through which a pump of horse-power H can deliver G gallons per second is d inches, where

$$d=\mathrm{G}^{\frac{3}{4}}\times(0\cdot41\mathrm{H})^{-\frac{1}{4}}.$$

Find d if $\mathrm{G}=250$, $\mathrm{H}=12\cdot5$.

17. If $p\,.\,v^{1\cdot4}=13\cdot72$, find p when $v=0\cdot8$.

18. Find the value of $\frac{0\cdot022v^2l}{gd^{1\cdot2}}$ if $v=8\cdot55$, $l=54\cdot6$, $g=32\cdot2$, $d=2\cdot66$.

19. Find the value of $35\cdot2\times e^{\mu\pi}$, if $e=2\cdot718$, $\mu=0\cdot465$.

20. Find the value of $10^4\times3\cdot6\times d^{0\cdot7}\times t^{0\cdot6}$ if $d=0\cdot9$, $t=0\cdot012$.

21. Find the value of x if $13x=\frac{a}{b}-\frac{1}{9}\log c$, when $a=343$, $b=51\cdot2$, $c=0\cdot426$.

22. A sum of money invested at r per cent. per annum compound interest is doubled after n years, where

$$n=\log 2\div\log\left(1+\frac{r}{100}\right).$$

Find n if $r=2\frac{1}{2}$.

23. Find θ from the formula, $\mu\theta=2\cdot303\log\left(\frac{\mathrm{P}}{\mathrm{W}}\right)$, if $\mu=0\cdot65$, $\mathrm{P}=27\cdot3$, $\mathrm{W}=16$.

24. If £P is set aside at the end of each year and accumulates at r per cent. per annum compound interest, the amount after n years is £A where

$$n=\log\left(\frac{\mathrm{A}r}{100\mathrm{P}}+1\right)\div\log\left(1+\frac{r}{100}\right).$$

Find n to the nearest integer if $\mathrm{P}=150$, $\mathrm{A}=1720$, $r=6$.

[*For additional examples, see Appendix, Ex. T. 1, p.* 166.]

CHAPTER II

VARIATION

Functions of One Variable

Direct Proportion. Suppose a train is running at a constant speed of 48 miles per hour.

In 5 minutes it travels 4 miles; in 10 minutes, 8 miles; in 15 minutes, 12 miles; and so on.

If the time is doubled, the corresponding distance is also doubled; if the time is halved, the distance is halved; etc.

We therefore say that the distance is *directly proportional* to the time, or that the distance *varies directly* as the time.

Suppose that in x minutes the train travels y miles, then

$$y=\tfrac{4}{5}x.$$

The corresponding travel graph is represented by OA *in* Fig. 2; **it is a straight line through the origin.**

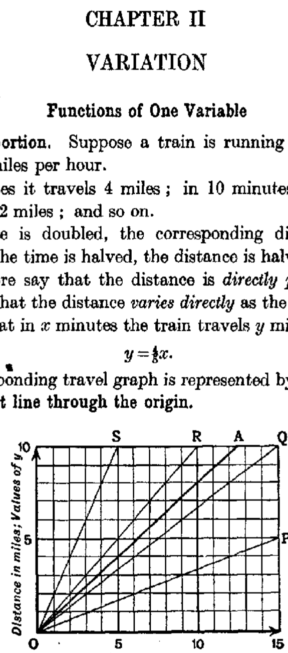


FIG. 2.

Fig. 2 contains also the travel graphs OP, OQ, OR of three other trains and the travel graph OS of an aeroplane. The reader should interpret each of these graphs and state (i) what speed each represents, (ii) what relation connects x and y.

For each graph in turn, we see that

$$\frac{y}{x}=c \text{ or } y=cx,$$

where c has some definite numerical value which measures the *slope* of the graph.

Thus for OA, $c=\frac{4}{5}$; and for OP, $c=\frac{1}{3}$; and for OS, $c=2$; etc.

If, then, the result of plotting values of y against values of x, gives a *straight line through the origin*, it follows that y varies directly as x; this is often written $y \propto x$, and we then know that x and y are connected by the relation, $\frac{y}{x}=c$ or $y=cx$, where c is constant.

Other Forms of Direct Variation. Open cubical tin boxes (*i.e.* having no lid) are made from tin sheeting in various sizes. Consider the relation between the area of tin sheeting used and the height of the manufactured box.

If the box is 3 in. high, the area of each face is 9 sq. in.; but there are 5 faces, $\therefore$ the area of sheeting required is 45 sq. in.

If we repeat this argument for various heights, we obtain the following table:

height in inches h -	2	3	4	5	6	7	8
area in sq. inches A -	20	45	80	125	180	245	320

It is obvious that the effect of doubling h is not the doubling of A; if we plot values of A against values of h, we shall not obtain a straight line through the origin, because $\frac{A}{h}$ is not constant. In other words, A does not vary directly as h.

Suppose, however, that we tabulate corresponding values of A and h^2.

A - -	20	45	80	125	180	245	320
h^2 -	4	9	16	25	36	49	64

It is evident that $\frac{A}{h^2}$ is constant; and we have

$$\frac{A}{h^2}=5 \quad \text{or} \quad A=5h^2.$$

We then say that A *varies directly as the square of h* or, merely, A varies as the square of h; and we write it, $A \propto h^2$.

The reader should now plot the values of A against the values of h^2, and verify the fact that the graph is a *straight line through the origin*.

If $A \propto h^2$, we see that if h is doubled or trebled, the corresponding value of A is found by multiplying by 4 or by 9, respectively; similarly if h is divided by 10, the corresponding value of A is obtained by dividing by 100.

Thus from $A = 245$ for $h = 7$ we deduce $A = 2{\cdot}45$ for $h = 0{\cdot}7$.

Example 1. The time of a complete oscillation of a simple pendulum varies as the square root of the length of the pendulum. It is found that a pendulum 9 feet long makes a complete oscillation in 3·3 seconds. What is the time of oscillation for a pendulum 4 feet long? Find also a general formula.

Suppose that the time of oscillation of a pendulum l feet long is t seconds.

Then we are given that $t \propto \sqrt{l}$.

We can now use any one of the following methods:

(i) $t = c \,.\, \sqrt{l}$ where c is constant.

But $t = 3{\cdot}3$ if $l = 9$, $\therefore\ 3{\cdot}3 = c\,.\sqrt{9} = 3c$.

$$\therefore\ c = 1{\cdot}1;$$
$$\therefore\ \mathbf{t = 1{\cdot}1\sqrt{l}}.$$

This is the required general formula.

If $l = 4$, $t = 1{\cdot}1\sqrt{4} = 1{\cdot}1 \times 2 = 2{\cdot}2$;

$\therefore$ for a pendulum 4 ft. long, the time of oscillation is **2·2 sec.**

(ii) Since t varies as $\sqrt{l}$, the value of $\frac{t}{\sqrt{l}}$ remains unchanged when l and t vary.

But $t = 3{\cdot}3$ when $l = 9$;

$$\therefore\ \frac{t}{\sqrt{l}} = \frac{3{\cdot}3}{\sqrt{9}} = \frac{3{\cdot}3}{3} = 1{\cdot}1.$$

Also when $l = 4$, $\frac{t}{\sqrt{4}} = 1{\cdot}1$, $\quad \therefore\ t = 1{\cdot}1 \times 2 = 2{\cdot}2$.

(iii) Since t is directly proportional to $\sqrt{l}$, we may also write

$$\frac{t}{3{\cdot}3} = \frac{\sqrt{l}}{\sqrt{9}}.$$

The three methods given in Example 1 show the different ways of expressing direct variation.

If y_1, y_2, y_3, etc., are values of y which correspond to the values x_1, x_2, x_3, etc., of x, and if $y \propto x$, then either

(i) $y_1 = cx_1,\ y_2 = cx_2,\ y_3 = cx_3$, etc.

or (ii) $\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} =$ etc.

or (iii) $\frac{y_1}{y_2} = \frac{x_1}{x_2},\ \frac{y_1}{y_3} = \frac{x_1}{x_3}$, etc.

EXERCISE II. a

1. At a certain time of day, the lengths of the shadows of objects of different heights are measured :

Height of object in feet -	h	15	30	60	75
Length of shadow in feet -	l	18	36	72	90

(i) What kind of relation would you expect to find between h and l ?

(ii) *Write down* the height of an object if the length of its shadow is 180 ft. ; 6 ft. ; 36×7 ft. ; 7·2 ft.

(iii) What kind of graph is obtained by plotting values of h against values of l ? What relation connects h and l ?

2. The weights of various square pieces of tin cut from the same sheet are as follows :

Side of square in cm. -	s	1	2	2·5	4	10
Weight of piece in gm. -	w	0·4	1·6	2·5	6·4	40

(i) What kind of relation would you expect to find between w and s ?

(ii) *Write down* the weight of a square piece if the length of its edge is 20 cm. ; 25 cm. ; $\frac{1}{2}$ cm. ; 3 cm.

(iii) Plot values of w against values of s^2, and then obtain a general formula.

3. The capacities of some water cans of the same shape are as follows :

Breadth of can in inches - -	b	4	5	6	8
Capacity of can in pints - -	P	3·2	6·25	10·8	25·6

(i) What kind of relation would you expect to find between P and b ?

(ii) *Write down* the capacity of a can, 10 in. broad ; 3 in. broad.

(iii) Obtain the formula connecting P and b.

4. If the area of a circle of radius r in. is A sq. in., then A varies as the square of r.

How is the value of A altered (i) by doubling r ; (ii) by halving r ; (iii) by increasing r by 10 per cent. ?

5. If the volume of a sphere of radius r in. is V cu. in., then V varies as the cube of r.

(i) How is the value of V altered (a) by multiplying the radius by 10; (b) by dividing the radius by 4?

(ii) The volume of a sphere of radius 1 in. is 4·19 cu. in., approximately; what is the volume of a sphere of radius 2 in.?

(iii) What is the increase per cent. in the volume, if the radius increases by 20 per cent.?

6. (i) Cloth of standard width is sold by the yard. If x yards of cloth cost y shillings, what kind of relation connects x and y?

(ii) Carpets of the same shape and material are sold in various sizes by the sq. yard. If a carpet x yards long costs y shillings, what kind of relation connects x and y?

(iii) A drug is sold in bottles of the same shape, but various sizes. If the value of the drug in a full bottle x inches high is y shillings, what kind of relation connects x and y?

7. (i) Silver circular discs are made up, all of the same thickness, so as to weigh 1 oz., 2 oz., 3 oz., 4 oz., etc. If the disc which weighs w oz. is of radius y inches, what kind of relation expresses y in terms of w?

(ii) Brass weights for a weighing machine are all of the same shape and are made up to weigh $\frac{1}{2}$ oz., 1 oz., 2 oz., 4 oz., etc. If the w oz. weight is y inches high, what kind of relation expresses y in terms of w?

8. Show that the values of x, y in the given table satisfy the law, $y \propto x^2$.
What is y (i) if $x=40$; (ii) if $x=3$?
What is x if $y=9$?

x -	2	4	8
y -	1	4	16

9. Show that the values of x, y in the given table satisfy the law, $y \propto \sqrt{x}$.
What is y (i) if $x=100$; (ii) if $x=36$?
What is x if $y=1{\cdot}4$?

x -	1	4	25
y -	0·2	0·4	1

10. Copy and fill in the following table, where $y \propto x^3$.

x -	0·5	1	2	3	4	5	20
y -		0·4				50	

11. If $y \propto x^2$ and if $y=2$ when $x=5$, *write down* (i) the value of y when $x=15$, (ii) the value of x when $y=200$.

12. If y varies as the cube of x, and if $y=3$ when $x=2$, *write down* (i) the value of y when $x=20$, (ii) the value of x when $y=\frac{3}{8}$.

13. If y varies as the square root of x, and if $y=5$ when $x=10$, *write down* (i) the value of y when $x=40$, (ii) the value of x when $y=15$.

14. Write down, *without* simplifying, the value of y missing from the given table,

x	5	7
y	3	

(i) if y varies as x;
(ii) if y varies as x^2;
(iii) if y varies as $\sqrt{x}$; (iv) if y varies as x^3.

15. It will be seen that the values of x in the following table have been chosen so that each exceeds the previous one by 50 per cent:

x - -	80	120	180	270
y - -	12			

Write down, *without* simplifying, the values of y missing from the table,

(i) if y varies as x;
(ii) if y varies as the fourth power of x;
(iii) if y varies as the cube root of x.

16. Two trays are the same shape. The cost of the gold for gilding the larger, which is 2 feet long, is £2; what is the cost for the smaller, 18 inches long? [The cost varies as the area.]

17. Copy, complete and interpret the following:

(i) if $A \propto r^2$, then $r \propto$;
(ii) if $V \propto r^3$, then $r \propto$;
(iii) if $t \propto \sqrt{l}$, then $l \propto$.

18. If $S \propto r^2$ and $V \propto r^3$, what is the effect on S and V if r is doubled? How does V alter if S becomes 100 times as large? What kind of relation expresses V in terms of S?

19. Plaster-cast models are made of a statue. A model which is 15 cm. high weighs 50 gm.; what will be the weight of a model 24 cm. high?

20. The distance of the horizon varies as the square root of the height above sea level. At a height of 24 feet, the distance is 6 miles. What is the distance at a height of 54 feet?

21. If a light beam is supported at each end and carries a fixed load at its mid-point, the sag at the middle varies as the cube of the length of the beam. For a beam 5 ft. long the sag is $1\frac{1}{4}$ in.; what is the sag if the beam is 6 ft. long?

22. How must the radius of a sphere be altered in order to double (i) the surface, (ii) the volume of the sphere ?

23. The tension of a wire of given material varies as the square of its length, if the musical note produced remains unchanged. A wire 10 in. long is at a tension of 12 lb. ; to what tension should a similar wire 15 in. long be strained to give the same note ?

24. If a planet whose mean distance from the Sun is d miles takes t years (*i.e.* Earth-years) for one revolution round the Sun, Kepler's third law states that t varies as $d^{\frac{3}{2}}$. Taking the mean distances of the Earth and Jupiter from the Sun as 93 million miles and 480 million miles, find in terms of Earth-years the length of a year on Jupiter.

25. A year on Mars is approximately 1·9 Earth-years ; use the data of No. 24 to find the mean distance of Mars from the Sun.

26. A solid sphere of radius 4 in. weighs 32 lb. Find the weight of a shell of the same material whose internal and external radii are 3 in. and 5 in.

Inverse Variation

A number of rectangular mats are made, of different shapes but all of the same area, 36 sq. in. ; their dimensions may be tabulated as follows :

Breadth, inches	x	2	3	4	5	6	8	9	10
Length, inches	y	18	12	9	7·2	6	4·5	4	3·6

As x increases, y decreases ; and in particular the effect of doubling x is to halve y, the effect of trebling x is to divide y by 3, etc. The general formula connecting x and y is

$$y=\frac{36}{x}=36\times\frac{1}{x},$$

and we say that y *varies inversely* as x, and we write $y\propto\frac{1}{x}$.

If we plot the values in the table, we obtain part of the graph of $y=\frac{36}{x}$ (see Fig. 3). This graph may be used to obtain scale-drawings of any number of mats belonging to the set, area 36 sq. in., as follows :

Take any point P on the graph ; from it draw lines parallel to the axes Ox, Oy meeting Oy, Ox in M, N. Since $\text{PN}=\frac{36}{\text{ON}}$, the area

of the rectangle ONPM is 36 units, and therefore ONPM represents on the selected scale one mat of the set. As P moves along the curve, the rectangle ONPM shows every possible form of mat of the set. Conversely, if you take any number of mats of the set,

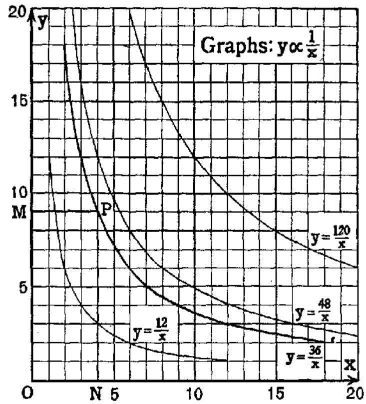


FIG. 3.

and arrange them so that two edges of each mat lie along Ox and Oy, then the corners opposite O mark out the curve. This suggests that if 36 is replaced by some other constant, the general form of the graph will be unaltered. Fig. 3 shows also parts of the graphs of $y=\frac{12}{x}$, $y=\frac{48}{x}$ and $y=\frac{120}{x}$ to illustrate this fact.

Actually, these curves are *similar*, although at a first glance they may not appear to be; this fact may be tested as follows: Draw any line OKPQR through O cutting the four curves at K, P, Q, R; draw *any other line* through O cutting the curves at k, p, q, r; then it will be found that the chords Kk, Pp, Qq, Rr are all parallel; actually in this figure, OK $=\frac{1}{2}$OQ, O$k=\frac{1}{2}$Oq.

If y varies inversely as x, then y varies directly as $\frac{1}{x}$; therefore if values of y are plotted against values of $\frac{1}{x}$, the graph will be a *straight line through the origin*. The following example is suggested for class use to illustrate the practical application of this fact.

Example 2. The pressure of a given mass of gas, kept at constant temperature, is measured for various volumes (v cu. cm.) by finding the height of a column of mercury (p cm.) which (as in a barometer) exerts the same pressure. The following observations are recorded :

v - - -	50	40	35	30	20
p - - -	77	102	111	135	197

Test whether these measurements obey Boyle's law, $p \propto \frac{1}{v}$, and, if they do so, find the relation between p and v, allowing for probable experimental errors.

First, tabulate corresponding values of $\frac{1}{v}$ and p.

$\frac{1}{v}$ - -	0·0200	0·0250	0·0286	0·0333	0·0500
p - -	77	102	111	135	197

Each member of the class should now plot values of p against values of $\frac{1}{v}$ on as large a scale as possible, but with each axis graduated from 0 *so that the origin* O *appears on the paper.*

If the observations were exact and did obey Boyle's law, the plotted points would lie exactly on a straight line through the origin. But some experimental errors are inevitable ; we do not therefore expect to find the points exactly on a straight line ; but we regard the test as satisfied if we can draw from the origin a straight line which passes very close to each of the plotted points, above some of them and below others. To find this "best-fit" line, take a piece of black thread, hold it so that it passes through O and swing it round O till it appears to pass evenly between the points. Note some point through which the thread passes and then, using a ruler, join this point to O.

Read off a pair of corresponding values of $\frac{1}{v}$ and p on the best-fit line, just drawn, and complete the equations,

$$\frac{p}{\frac{1}{v}} = \ldots\ldots ; \quad \therefore\ p = \frac{\ldots}{v}.$$

Other Forms of Inverse Variation

If y varies as $\frac{1}{x^2}$, we say that y varies inversely as the square of x.

If y varies as $\frac{1}{\sqrt{x}}$, we say that y varies inversely as the square root of x.

And so on.

To test whether $y \propto \frac{1}{x^2}$, we plot values of y against values of $\frac{1}{x^2}$, and see whether the plotted points lie on a straight line through the origin; similarly for other forms of inverse variation.

If $y \propto \frac{1}{x^2}$, then $y \div \frac{1}{x^2}$ is constant; this may be written either as $yx^2 = c$ or as $y = \frac{c}{x^2}$, where c is constant.

Similarly, if $y \propto \frac{1}{\sqrt{x}}$, then $y\sqrt{x} = c$ or $y = \frac{c}{\sqrt{x}}$, where c is constant.

Example 3. The number of beats per minute of a pendulum varies inversely as the square root of its length. A pendulum 9 feet long makes 36 beats per minute; how many beats per minute does a pendulum 4 feet long make?

Suppose a pendulum l feet long makes n beats per minute.

Either of the following methods may be used:

(i) When $l = 9$, $n = 36$; but $n \propto \frac{1}{\sqrt{l}}$;

$$\therefore \frac{n}{36} = \frac{\frac{1}{\sqrt{l}}}{\frac{1}{\sqrt{9}}} = \frac{3}{\sqrt{l}}.$$

$$\therefore \text{ if } l = 4, \ \frac{n}{36} = \tfrac{3}{2}; \quad \therefore n = \tfrac{3}{2} \times 36 = 54.$$

(ii) Since $n \propto \frac{1}{\sqrt{l}}$, $n\sqrt{l}$ is constant.

But $n = 36$ when $l = 9$, $\quad \therefore n\sqrt{l} = 36\sqrt{9} = 36 \times 3 = 108$;

$$\therefore \text{ if } l = 4, \ n\sqrt{4} = 108; \quad \therefore 2n = 108;$$

$$\therefore n = 54.$$

$\therefore$ a pendulum 4 feet long makes 54 beats a minute.

EXERCISE II. b

1. Copy and complete the following table, which shows the time taken to travel 24 miles at various speeds :

Speed in m.p.h. -	v	$1\frac{1}{2}$	2	3	4	5	6	8	12
Time in hours -	t			8					

(i) How is the value of t altered (*a*) if v is doubled, (*b*) if v is trebled, (*c*) if v is divided by 4.

(ii) How does t vary with v ?

(iii) What equation gives t in terms of v ?

(iv) Draw a rough figure (on plain paper) to show the shape of the graph obtained (*a*) if values of t are plotted against values of v, (*b*) if values of t are plotted against values of $\frac{1}{v}$.

(v) Draw a rough figure (on plain paper) to show the shape of the graph obtained by plotting values of t against values of v for a journey of 36 miles. Repeat, with the same axes, graphs for journeys of 48 miles, 60 miles, 120 miles.

2. Rectangular blocks of cast iron, of fixed weight 500 gm. and of equal density, are made up in various shapes ; the base of each block is square. Copy and complete the following table :

Height of block in cm. -	h			8	4·5		2	0·5
Side of square base in cm. - - - -	x	1·5	2	3		5	6	

(i) How is the value of h altered (*a*) if x is doubled, (*b*) if x is trebled, (*c*) if x is divided by 6 ?

(ii) What equation gives h in terms of x ? How does h vary with x ?

(iii) What equation gives x in terms of h ? How does x vary with h ?

3. If a pendulum l feet long makes n beats per minute, corresponding values of n and l are as follows :

l - -	$2\frac{1}{4}$	4	9	16	25	36
n - -	72	54	36	27	...	18

(i) Test by numerical calculation that n varies inversely as the square root of l, and find a formula connecting n and l.

(ii) Complete the table. Find also the length of a second's pendulum, that is a pendulum which makes one beat per second.

(iii) If a pendulum clock is gaining, should the pendulum be shortened or lengthened ?

(iv) How is the number of beats per minute altered if the length of the pendulum (a) is multiplied by 9, (b) is halved ?

4. A body (weight w gm.) is attached to one end of a given piece of string, held at the other end, and is whirled round in a horizontal circle. If the string breaks when the body is making n revolutions per second, it can be proved that w varies inversely as the square of n, the length and strength of the string remaining fixed.

It is found that for a body of weight 200 gm., the string breaks at 6 revs. per sec. Under what load will this string break at 12 revs. per sec., and what is the maximum rate for a body of weight 8 gm. ?

Find also a general formula for w in terms of n, and state how n varies with w.

5. What kind of relation connects the number of days, y, that the food in a besieged city will last, with the number of people, x, in the city ?

6. Cylindrical jam jars of various shapes hold equal amounts of jam. What kind of relation connects the height, h in., of the jar with its base radius, r in. ?

7. What kind of relation connects the density, w gm. per cu. cm., of the air inside a given soap bubble with the radius r cm. of the bubble, if it expands or contracts ?

What kind of relation expresses r in terms of w ?

8. If $y \propto \frac{1}{x}$, and if $y=12$ when $x=6$, find the value (i) of y when $x=8$, (ii) of x when $y=\frac{1}{2}$.

9. If $y \propto \frac{1}{x^2}$, and if $y=9$ when $x=10$, find the value (i) of y when $x=6$, (ii) of x when $y=4$.

10. If y varies inversely as $\sqrt{x}$ and if $y=5$ when $x=16$, find the value (i) of y when $x=100$, (ii) of x when $y=60$.

Try to discover simple laws of variation, expressing y in terms of x, to fit the following, Nos. 11-14 :

11.

x - -	2	3	4	6	20
y - -	30	20	15	10	3

12.

x	2	3	4	6	20
y	36	16	9	4	0·36

13.

x	2	3	4	6	20
y	20	45	80	180	2000

14.

x	4	9	16	25	100
y	60	40	30	24	12

15. If $y \propto \frac{1}{x}$, and if $y=7$ when $x=9$, *write down* (without simplifying) (i) the value of y when $x=11$; (ii) the value of x when $y=13$.

16. If $y \propto \frac{1}{\sqrt{x}}$ and if $y=5$ when $x=10$, *write down* (without simplifying) the value of y when $x=7$.

Write down also how x varies with y, and then write down the value of x when $y=6$.

17. The weight of a given body, as measured by a spring balance, varies inversely as the square of its distance from the centre of the Earth, if it is above the Earth's surface. A body weighs 10 lb. when on the ground; what will it weigh if it is 1000 miles above the Earth's surface? [Radius of Earth$\simeq$4000 miles.]

18. With the data of No. 17, find the height to which a body must be raised to reduce its weight by 25 per cent.

19. For wires of fixed length and material, the electrical resistance varies inversely as the square of the diameter of the cross-section. If the resistance is 0·5 ohms when the diameter is 1·2 cm., find the resistance of a wire of the same length and composition, 1·5 cm. in diameter.

20. Two unlike magnetic poles attract one another with a force which varies inversely as the square of the distance between them. At 6 cm. apart, the force of attraction is found to be 5·4 gm. wt.; what will the force be if they are placed 9 cm. apart?

What is the percentage change in the force, if the distance is increased by 25 per cent.?

Functions of Two or More Variables

Suppose we have a number of cylindrical metal bolts and we wish to compare their weights.

The weight of any particular bolt depends on several quantities which can alter independently, the radius of its circular cross-section r in., the length of the bolt l in., and the density of the material composing it, ρ lb. per cu. in.

Since the volume of the bolt is $\pi r^2 l$ cu. in., the weight W lb. of the bolt is given by the formula

$$W = \pi\rho r^2 l.$$

Suppose we have a number of cast-iron bolts (density 0·26 lb. per cu. in.), each of radius 0·75 in., then

$$W = \pi \times 0{\cdot}26 \times (0{\cdot}75)^2 l \simeq 0{\cdot}46 l.$$

This relation can be described by saying that if the bolts are all of the same material and of equal cross-section, then $W \propto l$ or $\frac{W}{l}$ is constant; the value of this constant can be found, as above, if we know the material and the cross-section.

Similarly, suppose we have a number of copper bolts (density 0·31 lb. per cu. in.), each of length 3 in., then

$$W = \pi \times 0{\cdot}31 \times r^2 \times 3 \simeq 2{\cdot}9 r^2;$$

and we say that, if the bolts are all of the same material and of equal length, then $W \propto r^2$ or $\frac{W}{r^2}$ is constant; and the value of the constant can be found as above.

Similarly, if all the bolts are the same size, but of various materials,

$$W \propto \rho \text{ or } \frac{W}{\rho} \text{ is constant, } l \text{ and } r \text{ being fixed.}$$

Now the variables ρ, r, l, or powers of these variables, are *factors* of the expression for W, and the remaining factor is a constant. We therefore say that W varies directly as ρ and as l and as the square of r. This is called *joint variation.*

Example 4. If the density of a cylindrical bolt is increased by 20 per cent., and the radius is decreased by 25 per cent., and the length is increased by 60 per cent., find the percentage increase in the weight.

If the original density is ρ lb. per cu. in. and the original radius

is r in. and the original length is l in., then the original weight W lb. is given by $W = \pi\rho r^2 l$.

$\therefore$ the new weight W_1 lb. is given by

$$W_1 = \pi\left(\frac{120\rho}{100}\right) \times \left(\frac{75r}{100}\right)^2 \times \left(\frac{160l}{100}\right);$$

$$\therefore \frac{W_1}{W} = \tfrac{6}{5} \times (\tfrac{3}{4})^2 \times \tfrac{8}{5} = \tfrac{27}{25} = \tfrac{108}{100}.$$

$\therefore$ W_1 exceeds W by 8 per cent.

It will be seen that this argument can be expressed more shortly, as follows:

Since $$W \propto \rho r^2 l,$$

if the density is multiplied by $\frac{120}{100} = \frac{6}{5}$, and if the radius is multiplied by $\frac{75}{100} = \frac{3}{4}$, and if the length is multiplied by $\frac{160}{100} = \frac{8}{5}$, then the weight is multiplied by $\frac{6}{5} \times (\frac{3}{4})^2 \times \frac{8}{5} = \frac{27}{25} = \frac{108}{100}$;

$\therefore$ the weight increases by 8 per cent.

Suppose next we wish to compare the *densities* of a number of cylindrical metal bolts. Take the formula obtained above, $W = \pi\rho r^2 l$, and make ρ the subject. Then

$$\rho = \frac{1}{\pi} \cdot \frac{W}{r^2 l}.$$

Here ρ is expressed as a function of the three variables W, r, l; but further the *factors* of this expression for ρ are powers of these variables, and the remaining factor is a constant, since

$$\rho = \frac{1}{\pi} \times W \times r^{-2} \times l^{-1}.$$

Therefore we say that ρ varies directly as W and inversely as the square of r and inversely as l. This is another type of joint variation.

Example 5. If the radius of a cylindrical bolt is increased by 25 per cent., and if the length is decreased by 70 per cent., find the percentage change in the density, if the weight is reduced by 40 per cent.

With the previous notation, $\rho \propto \dfrac{W}{r^2 l}$.

If W is multiplied by $\frac{60}{100} = \frac{3}{5}$, and if r is multiplied by $\frac{125}{100} = \frac{5}{4}$, and if l is multiplied by $\frac{30}{100} = \frac{3}{10}$, then the density is multiplied by

$$\frac{\frac{3}{5}}{(\frac{5}{4})^2 \times (\frac{3}{10})} = \tfrac{3}{5} \times \tfrac{16}{25} \times \tfrac{10}{3} = \tfrac{32}{25} = \tfrac{128}{100};$$

$\therefore$ the density increases by 28 per cent.

Example 6. The volume of a given mass of gas varies directly as the absolute temperature and inversely as the pressure.

At absolute temperature 288° and at pressure 825 mm., the volume is 450 c.c.; what is its volume at absolute temperature 320° and at pressure 750 mm.?

If the volume is v c.c. at absolute temperature T° and at pressure p mm., then $v \propto \frac{\mathrm{T}}{p}$.

When $v=450$, $\mathrm{T}=288$, $p=825$;

$\therefore$ the value of v when $\mathrm{T}=320$ and $p=750$ is given by

$$\frac{v}{450}=\frac{320}{750}\div\frac{288}{825}=\frac{320\times 825}{750\times 288};$$

$$\therefore\ v=\frac{320\times 825}{750\times 288}\times 450=550.$$

$\therefore$ the volume is **550 c.c.**

The following method is easier but a little longer:

Since $v\propto\frac{\mathrm{T}}{p}$, $v=k\times\frac{\mathrm{T}}{p}$ where k is constant.

But $v=450$ when $\mathrm{T}=288$ and $p=825$,

$$\therefore\ 450=k\times\frac{288}{825};\quad \therefore\ k=\frac{450\times 825}{288};$$

$$\therefore\ v=\frac{450\times 825}{288}\times\frac{\mathrm{T}}{p};$$

$\therefore$ when $\mathrm{T}=320$ and $p=750$, $v=\frac{450\times 825}{288}\times\frac{320}{750}=550$, as before.

If a function of a variable consists of **two or more terms**, it does not vary directly or inversely as any power of that variable, although the separate terms may do so.

Example 7. If a train is running along the level, the distance in which it can be stopped varies *partly* as the velocity and *partly* as the square of the velocity. Express this statement in symbols.

Suppose that the train can be stopped in s yards when running at v miles per hour.

If s is expressed in terms of v, the function is formed of two parts; one of them varies directly as v, call it av; the other varies directly as v^2, call it bv^2. Then

$$\mathbf{s=av+bv^2}$$

where a and b are constants.

It is untrue to say that s "varies as" v or as any power of v, although of course the value of s changes when the value of v changes. But the phrase "s varies as some power of v" is reserved for relations of the form, $\frac{s}{v^n}=$constant. For relations such as $s=av+bv^2$, the phrase "partly varies" is used.

Similarly, the relation $s=a+bv^2$, where a, b are constants, is sometimes described by the phrase, "s is *partly constant and partly varies* as the square of v."

Example 8. The volume of a segment of a sphere varies partly as the radius and the square of the height jointly, and partly as the cube of the height. Find the formula for the volume V cu. in. of a segment of a sphere of radius r in., height of segment h in., given that the volume of a sphere of radius r in. is $\frac{4}{3}\pi r^3$ cu. in.

If V is expressed as a function of r and h, there are two terms; one of them $\propto r\times h^2$, the other $\propto h^3$.

$$\therefore\ V=arh^2+bh^3,$$

where a and b are constant.

Now a hemisphere is a segment such that $h=r$.

$$\therefore\ \text{if } h=r,\ V=\tfrac{2}{3}\pi r^3;\quad \therefore\ \tfrac{2}{3}\pi r^3=ar^3+br^3.$$

$$\therefore\ a+b=\frac{2\pi}{3}.$$

Also the complete sphere is a segment such that $h=2r$;

$$\therefore\ \text{if } h=2r,\ V=\tfrac{4}{3}\pi r^3;\quad \therefore\ \tfrac{4}{3}\pi r^3=ar(2r)^2+(b2r)^3;$$

$$\therefore\ 4a+8b=\frac{4\pi}{3}.$$

Solving for a, b, we have $a=\pi$, $b=-\frac{\pi}{3}$.

$$\therefore\ V=\pi rh^2-\frac{\pi}{3}h^3.$$

EXERCISE II. c

1. A rectangular brass slab has a square base, side x in., and is h in. high; its weight W lb. is given by the formula $W = \frac{3}{10}x^2h$.

   (i) How is W altered if x is trebled and h remains constant?

   (ii) How is W altered if x is trebled and h is halved?

   (iii) Complete the sentence: W varies ....

   (iv) What kind of variation expresses h in terms of W, x?

   (v) What kind of variation expresses x in terms of W, h?

   (vi) What is the effect on h if W is trebled and x is multiplied by 4?

   (vii) What is the effect on x if W is divided by 4 and h is multiplied by 100?

Express the following statements by symbols, Nos. 2-7:

2. The volume of a cone varies directly as the height and the square of the base-radius. (V, h, r.)

3. The force necessary to stop a tramcar in a certain distance varies directly as its weight and the square of its velocity and inversely as the distance. (F, W, v, d.)

4. The time taken to repair a road varies directly as the length of the road and as the square root of its breadth and inversely as the number of men employed. (t, l, b, n.)

5. The electrical resistance of a wire varies directly as its length and inversely as the square of its diameter. (R, l, d.)

6. The cost of the fuel consumed by a coasting steamer between two ports of call varies as the cube of the distance and inversely as the square of the time. (C, d, t.)

7. The resistance to the motion of a train of given weight is partly constant and partly varies as the square of the velocity.

8. Copy and fill in the given table for the values of a function Z,

   (i) if Z varies directly as x and inversely as y;

   (ii) if Z varies directly as x^3 and inversely as y^2.

   [*All working to be done mentally.*]

		Values of x.		
		3	6	12
Values of y.	2			
	10		7	
	50			

9. If W varies directly as x and inversely as y, and if $W=9$ when $x=12$ and $y=2$, find W when $x=20$ and $y=15$.

10. If Z varies directly as the square of x and inversely as the square root of y, and if $Z=3$ when $x=6$, $y=16$, find Z when $x=12$, $y=25$.

11. A mass w lb. is fastened to one end of a string l feet long, and the other end is held in the hand. If the mass is now whirled round in a horizontal circle at speed v ft. per sec., the tension, T lb. weight, in the string varies directly as w and the square of v and inversely as l. If the mass of the body is $1\frac{1}{2}$ lb., if the string is 3 ft. long and if the speed is 40 ft. per sec., the tension is 25 lb. wt.; find the general formula.

12. The heat, h calories, developed in a wire varies directly as the time, t seconds, and as the square of the voltage, V volts, and inversely as the resistance, R ohms.

If $V=100$ and $R=50$, the heat developed is 48 calories per second. Find how much heat is developed in 1 minute in a wire of resistance 2 ohms, if the voltage is 2 volts.

13. A smooth plank is held at a slope with one end on the ground and objects slide down it. The time of descent varies directly as the length of the plank and inversely as the square root of the height of the top of the plank above the ground. For a plank 30 ft. long, with the upper end 25 ft. above the ground, the time of descent is 1·5 sec.; find the general formula.

14. The illumination of a small object by a lamp varies directly as the candle power and inversely as the square of its distance from the lamp. If an electric lamp of 32 candle power, fixed 6 ft. above a table, is replaced by a lamp of 18 candle power, how much must the new lamp be lowered to give the same illumination as before at the point of the table directly below the lamp?

15. If a wooden beam is supported at its ends, the central breaking load varies directly as the width and as the square of the thickness of the beam and inversely as the length of the beam. A specimen of pitch pine 1 in. wide, 1 in. thick, and 2 ft. long was found to break under a load of 200 lb. What is the breaking load for a beam of the same timber, 8 ft. long, 6 in. wide, 1 ft. thick? Answer to nearest ton.

16. The deflection at the centre of a girder, l ft. long, of given material with fixed ends under a uniformly distributed load, W tons, varies as $\frac{Wl^3}{I}$, where I is the moment of inertia about the centre. If the load is increased by 50 per cent. and the moment of inertia by 20 per cent., find what percentage change must be made in the length, in order that the deflection may be unaltered.

17. (i) Equal masses of copper are drawn out into wires of different lengths. Use the fact stated in No. 5 to prove that the electrical resistance varies as the square of the length.

(ii) If a number of copper wires are of equal lengths, find how the resistance varies with the weight.

18. Use the statement in No. 6 to find how the cost varies with the distance and the speed, assumed uniform.

19. If $\mathsf{P} \propto \frac{v^2}{x}$ and $x \propto vt$, find how P varies (i) with v and t, (ii) with x and t. [These results illustrate certain formulae in dynamics.]

20. The weight that can be carried by a cylindrical iron column varies directly as the fourth power of the diameter and inversely as the square of its length.

(i) If the length is doubled, how must the diameter be altered, if the same weight is to be carried ?

(ii) If a load W tons is to be carried by a cylindrical column l ft. long and of volume V cu. ft., find how W varies with l and V.

21. y is partly constant and partly varies as x. When x is 2, y is 16, and when x is 7, y is 31. Find a general formula for y in terms of x.

22. W varies partly as x and partly as the square of x. When x is 2, W is 14, and when x is 4, W is 44. Find W when x is 3.

23. The resistance to the motion of a car is partly constant and partly varies as the square of the velocity. When the car is travelling at 20 m.p.h., the resistance is 60 lb.-wt., and at 30 m.p.h., it is 85 lb.-wt. Find the resistance when the car is travelling at 40 m.p.h.

24. y varies partly as x and partly inversely as x. When x is 3, y is 10, and when x is 4, y is 11. Find the value of y when x is (i) 2, (ii) $2\frac{1}{2}$.

25. The distance in which a train can be stopped varies partly as the velocity and partly as the square of the velocity. When running at 20 m.p.h., the train can be stopped in 100 yd., and at 40 m.p.h. in 280 yd. Find the greatest speed at which the train can be travelling, if it can be just stopped in a quarter of a mile.

[*For additional examples, see Appendix, Ex. T.* 2, *p.* 167. *For a revision exercise on Ch. I-II, see Appendix, Ex. W.* 1, *p.* 145.]

CHAPTER III

SURDS

Rational and Irrational Numbers. It can be proved that numbers such as $\sqrt{2}$, $\sqrt[3]{10}$, π, etc., cannot be expressed exactly as the *ratio* of two integers: they are therefore called *irrational* numbers; whereas numbers which can be reduced to the form $\frac{p}{q}$, where p and q are integers, are called *rational*. It is of course possible to obtain rational numbers which approximate as closely as we please to any given irrational number; thus $\frac{22}{7}$ is an approximation for π and $\frac{314159}{100000}$ is a much closer approximation.

If the root of any rational number is not itself rational, it is called a *surd*, thus $\sqrt{(\frac{2}{3})}$ and $\sqrt[3]{7}$ are surds, but $\sqrt[4]{81}$ is not a surd because $\sqrt[4]{81}=3$.

Although a number such as 9 has two square roots, $+3$ and -3, we shall use the symbol $\sqrt{9}$ to denote the *positive* square root of 9, and in general $\sqrt{x}$ denotes the positive square root of x.

EXERCISE III. a

If $a=4$, $b=9$, $c=16$, $d=25$, evaluate the following pairs of expressions and note which pairs are unequal.

1. $\sqrt{a} \times \sqrt{b}$; $\sqrt{(ab)}$.
2. $\sqrt{b} + \sqrt{c}$; $\sqrt{(b+c)}$.
3. $\sqrt{d} - \sqrt{c}$; $\sqrt{(d-c)}$.
4. $\sqrt{b} \div \sqrt{d}$; $\sqrt{(b \div d)}$.
5. $\sqrt{(100a)}$; $10\sqrt{a}$.
6. $\frac{1}{\sqrt{b}}$; $\frac{\sqrt{b}}{b}$.
7. $(\sqrt{a}+\sqrt{b})^2$; $a+b+2\sqrt{(ab)}$.
8. $(\sqrt{b}+\sqrt{a})(\sqrt{b}-\sqrt{a})$; $b-a$.
9. $\sqrt{a}(\sqrt{b}+\sqrt{d})$; $\sqrt{(ab)}+\sqrt{(ad)}$.
10. $(\sqrt{c}-\sqrt{b})^2$; $c+b-2\sqrt{(cb)}$.

Operations with Surds. All the operations in Ex. III. a involve only rational numbers. They illustrate that, when $\sqrt{a}$ and $\sqrt{b}$ are rational,

$$\text{(i)}\ \sqrt{a} \times \sqrt{b} = \sqrt{(ab)}; \quad \text{(ii)}\ \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\left(\frac{a}{b}\right)}.$$

And that $\sqrt{a}+\sqrt{b}$ is *not equal* to $\sqrt{(a+b)}$, unless $a=0$ or $b=0$; and that $\sqrt{a}-\sqrt{b}$ is *not equal* to $\sqrt{(a-b)}$, unless $b=0$.

We shall now *assume* that the fundamental laws of algebra which govern operations with rational numbers hold also for irrational numbers. Consequently,

$$(\sqrt{a}\times\sqrt{b})^2=\sqrt{a}\times\sqrt{b}\times\sqrt{a}\times\sqrt{b}=\sqrt{a}\times\sqrt{a}\times\sqrt{b}\times\sqrt{b}$$
$$=a\times b=ab.$$

Now take the square root of each expression,

then $$\sqrt{\mathbf{a}}\times\sqrt{\mathbf{b}}=\sqrt{(\mathbf{ab})}.$$

Similarly, $$\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2=\frac{\sqrt{a}}{\sqrt{b}}\times\frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{a}\times\sqrt{a}}{\sqrt{b}\times\sqrt{b}}=\frac{a}{b}.$$

Take the square root of each expression,

then $$\frac{\sqrt{\mathbf{a}}}{\sqrt{\mathbf{b}}}=\sqrt{\left(\frac{\mathbf{a}}{\mathbf{b}}\right)}.$$

Example 1. Given that $\sqrt{3}\simeq 1{\cdot}732$, evaluate to three significant figures (i) $\sqrt{75}$; (ii) $\frac{2}{\sqrt{3}}$.

(i) $\sqrt{75}=\sqrt{(25\times 3)}=\sqrt{25}\times\sqrt{3}=5\sqrt{3}$
$\simeq 5\times 1{\cdot}732=8{\cdot}66.$

(ii) $$\frac{2}{\sqrt{3}}=\frac{2\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}=\frac{2\sqrt{3}}{3}\simeq\frac{2\times 1{\cdot}732}{3}\simeq 1{\cdot}15.$$

Note. The fact, $\sqrt{3}\times\sqrt{3}=3$, is the definition of $\sqrt{3}$, and it should be written down immediately, instead of saying

$$\sqrt{3}\times\sqrt{3}=\sqrt{9}=3.$$

Example 1 (ii) illustrates an important practical method. Fractions with surds in the denominator can often be dealt with more easily if replaced by equivalent fractions with rational denominators. This process is called "rationalising the denominator."

Example 2. Simplify (i) $\sqrt{15}\times\sqrt{12}$; (ii) $\frac{12}{\sqrt{18}}$.

(i) $\sqrt{15}\times\sqrt{12}=\sqrt{(15\times 12)}=\sqrt{(2^2\,.\,3^2\,.\,5)}$
$=2\,.\,3\,.\,\sqrt{5}=6\sqrt{5}.$

Or, proceed as follows:

$$\sqrt{15}\times\sqrt{12}=(\sqrt{3}\times\sqrt{5})\times(\sqrt{4}\times\sqrt{3})$$
$$=2\times\sqrt{3}\times\sqrt{3}\times\sqrt{5}=2\times 3\times\sqrt{5}=6\sqrt{5}.$$

(ii) $\frac{12}{\sqrt{18}}=\frac{12}{\sqrt{9}\times\sqrt{2}}=\frac{12}{3\sqrt{2}}$

$=\frac{4\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{4\sqrt{2}}{2}=2\sqrt{2}.$

EXERCISE III. b

Express the following surds so that the integer under the square root sign is as small as possible :

1. $\sqrt{8}$. 2. $\sqrt{12}$. 3. $\sqrt{90}$. 4. $\sqrt{18}$.
5. $\sqrt{48}$. 6. $\sqrt{50}$. 7. $\sqrt{20}$. 8. $\sqrt{27}$.
9. $\sqrt{80}$. 10. $\sqrt{28}$. 11. $\sqrt{540}$. 12. $\sqrt{720}$.

Express the following as the square roots of integers :

13. $2\sqrt{3}$. 14. $3\sqrt{2}$. 15. $5\sqrt{10}$. 16. $2\sqrt{7}$.
17. $3\sqrt{6}$. 18. $5\sqrt{5}$. 19. $7\sqrt{2}$. 20. $\frac{1}{2}\sqrt{48}$.

Express the following with rational denominators :

21. $\frac{1}{\sqrt{2}}$. 22. $\frac{4}{\sqrt{2}}$. 23. $\frac{2}{\sqrt{6}}$. 24. $\frac{10}{\sqrt{12}}$.
25. $\frac{6}{\sqrt{27}}$. 26. $\frac{15}{\sqrt{50}}$. 27. $\sqrt{(\frac{4}{3})}$. 28. $5\sqrt{(\frac{2}{5})}$.

Evaluate to 3 significant figures the following, given that

$$\sqrt{2}\simeq 1{\cdot}414, \quad \sqrt{5}\simeq 2{\cdot}236.$$

29. $\frac{10}{\sqrt{2}}$. 30. $\frac{1}{\sqrt{5}}$. 31. $\frac{1}{\sqrt{50}}$. 32. $\frac{5}{\sqrt{20}}$.

Simplify the following :

33. $\sqrt{2}\times\sqrt{8}$. 34. $\sqrt{6}\times\sqrt{3}$. 35. $\sqrt{10}\times\sqrt{20}$.
36. $3\sqrt{2}\times\sqrt{6}$. 37. $2\sqrt{5}\times5\sqrt{2}$. 38. $3\sqrt{8}\times2\sqrt{8}$.
39. $\sqrt{2}\times\sqrt{(\frac{1}{2})}$. 40. $\sqrt{2}\div\sqrt{(\frac{1}{2})}$. 41. $\sqrt{6}\div\sqrt{12}$.
42. $(\sqrt{3})^3$. 43. $2[\sqrt{(\frac{1}{2})}]^3$. 44. $\sqrt{6}\times\sqrt{8}\times\sqrt{12}$.

Reduction

Example 3. Simplify $\sqrt{8}+\sqrt{50}-6\sqrt{(\frac{1}{2})}$.

$\sqrt{8}=\sqrt{4}\times\sqrt{2}=2\sqrt{2}$; $\quad\sqrt{50}=\sqrt{25}\times\sqrt{2}=5\sqrt{2}$;

$6\sqrt{(\frac{1}{2})}=6\times\frac{1}{\sqrt{2}}=6\times\frac{\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{6\sqrt{2}}{2}=3\sqrt{2}$;

$\therefore$ expression $=2\sqrt{2}+5\sqrt{2}-3\sqrt{2}=4\sqrt{2}.$

Example 4. Expand $(\sqrt{3}-\sqrt{2})^2$.

Since $(a-b)^2=a^2-2ab+b^2$, we have

$$\begin{aligned}(\sqrt{3}-\sqrt{2})^2&=(\sqrt{3})^2-2(\sqrt{3})(\sqrt{2})+(\sqrt{2})^2\\&=3-2\sqrt{6}+2=5-2\sqrt{6}.\end{aligned}$$

Example 5. Expand $(5\sqrt{2}+4\sqrt{3})(5\sqrt{2}-4\sqrt{3})$.

Since $(a+b)(a-b)=a^2-b^2$, we have

$$\begin{aligned}(5\sqrt{2}+4\sqrt{3})(5\sqrt{2}-4\sqrt{3})&=(5\sqrt{2})^2-(4\sqrt{3})^2\\&=25\times2-16\times3=50-48\\&=2.\end{aligned}$$

Example 6. Express with a rational denominator, $\dfrac{\sqrt{3}}{3\sqrt{3}+4}$.

$$\begin{aligned}\frac{\sqrt{3}}{3\sqrt{3}+4}&=\frac{\sqrt{3}(3\sqrt{3}-4)}{(3\sqrt{3}+4)(3\sqrt{3}-4)}=\frac{3\sqrt{3}\times\sqrt{3}-4\sqrt{3}}{(3\sqrt{3})^2-4^2}\\&=\frac{9-4\sqrt{3}}{9\times3-16}=\frac{9-4\sqrt{3}}{27-16}\\&=\frac{9-4\sqrt{3}}{11}.\end{aligned}$$

EXERCISE III. c

Simplify the following:

1. $\sqrt{2}+\sqrt{8}$. 2. $\sqrt{12}-\sqrt{3}$. 3. $2\sqrt{20}-\sqrt{80}$.

4. $\sqrt{2}-\sqrt{(\frac{1}{2})}$. 5. $\sqrt{3}-\dfrac{1}{\sqrt{3}}$. 6. $\sqrt{(\frac{3}{2})}-\sqrt{(\frac{2}{3})}$.

7. $\sqrt{27}+\sqrt{12}-\sqrt{48}$. 8. $\sqrt{20}+2\sqrt{45}-\sqrt{125}$.

9. $6\sqrt{(\frac{1}{3})}+\sqrt{(\frac{4}{3})}-\dfrac{1}{\sqrt{3}}$. 10. $\sqrt{(\frac{4}{5})}+\sqrt{(\frac{1}{5})}-\dfrac{5}{\sqrt{20}}$.

11. $\sqrt{2}(1+\sqrt{2})$. 12. $\sqrt{6}(\sqrt{3}+\sqrt{2})$. 13. $2\sqrt{2}(\sqrt{8}+3\sqrt{2})$.

14. $(\sqrt{3}+2)(\sqrt{3}-1)$. 15. $(3+2\sqrt{2})(5-3\sqrt{2})$.

16. $(5\sqrt{6}-7\sqrt{2})(2\sqrt{6}+3\sqrt{2})$.

17. $(2\sqrt{10}-\sqrt{5})(2\sqrt{2}+1)$. 18. $(5\sqrt{2}-3)^2$.

19. $(5\sqrt{6}+3\sqrt{2})^2$. 20. $(3+\sqrt{7})(3-\sqrt{7})$.

21. $(2\sqrt{5}-5\sqrt{7})(2\sqrt{5}+5\sqrt{7})$. 22. $(\sqrt{x}-\sqrt{y})^2$.

23. $(\sqrt{c}+\sqrt{d})(\sqrt{c}-\sqrt{d})$.

Express the following with rational denominators:

24. $\dfrac{1}{\sqrt{2}+1}$. 25. $\dfrac{1}{3-\sqrt{5}}$. 26. $\dfrac{6}{\sqrt{3}-1}$.

27. $\frac{1}{\sqrt{5}-\sqrt{3}}$. 28. $\frac{1}{3\sqrt{2}-3}$. 29. $\frac{\sqrt{2}}{\sqrt{6}-\sqrt{2}}$.

30. $\frac{\sqrt{2}+1}{\sqrt{2}-1}$. 31. $\frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}$. 32. $\frac{3\sqrt{5}-2}{2\sqrt{5}-4}$.

Simplify the following :

33. $\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{3}-1}$. 34. $\frac{\sqrt{5}+2}{\sqrt{5}-2}-\frac{\sqrt{5}-2}{\sqrt{5}+2}$.

35. If $a=\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$, find, correct to 3 significant figures, the value of (i) $a-\frac{1}{a}$; (ii) $\frac{a-1}{a+1}$.

36. Solve (to 3 figures) $x+x\sqrt{3}=10$.

37. Find x in terms of a if $(x+a)^2=3x^2$.

38. Evaluate y^2+2y when $y=\sqrt{3}-1$.

39. The perimeter of an isosceles right-angled triangle is 10 cm.; find, correct to $\frac{1}{10}$ mm., the length of each of the equal sides.

40. In $\triangle ABC$, $AB=2$ in., $BC=1$ in., $\angle ABC=90°$; P is a point on AC such that $CP=1$ in.; Q is a point on AB such that $AQ=AP$. Write down expressions for the lengths of AC, AQ, BQ and prove that $AB \cdot BQ=AQ^2$.

[*For further practice, see Appendix, Ex. F.P.* 4, *Nos.* 1-22, *p.* 156.]

Example 7. Find the square roots of $21-6\sqrt{10}$.

The square of $\sqrt{x}-\sqrt{y}$ is $x+y-2\sqrt{(xy)}$.

If therefore we wish to express $21-6\sqrt{10}$ as a square, we first write it in the form $21-2\sqrt{90}$, and then try to find two numbers x and y such that

$$x+y=21; \quad xy=90.$$

Either by solving in the ordinary way (Part II, Ch. XV) or by inspection, we have

$$x=15, y=6 \text{ or } y=15, x=6.$$

$\therefore$ $(\sqrt{15}-\sqrt{6})^2$ and $(\sqrt{6}-\sqrt{15})^2$ each equal $21-6\sqrt{10}$.

$\therefore$ the square roots of $21-6\sqrt{10}$ are $\pm(\sqrt{15}-\sqrt{6})$.

Note. To find the square roots of $21+6\sqrt{10}$, we use the fact that the square of $\sqrt{x}+\sqrt{y}$ is $x+y+2\sqrt{(xy)}$; the square roots are obviously $\pm(\sqrt{15}+\sqrt{6})$.

Example 8. Solve $\sqrt{x}+\sqrt{(3x+1)}=3$.

We have $\qquad \sqrt{(3x+1)}=3-\sqrt{x}$.

Square each side ; $\therefore\ 3x+1=9+x-6\sqrt{x}$;

$$\therefore\ 2x-8=-6\sqrt{x}\,; \quad \therefore\ x-4=-3\sqrt{x}\,;$$

square each side ; $\therefore\ x^2-8x+16=9x$;

$$\therefore\ x^2-17x+16=0\,; \quad \therefore\ (x-1)(x-16)=0\,;$$

$$\therefore\ x=1 \text{ or } 16.$$

If $x=1$, $\sqrt{x}+\sqrt{(3x+1)}=\sqrt{1}+\sqrt{4}=1+2=3$.

If $x=16$, $\sqrt{x}+\sqrt{(3x+1)}=\sqrt{16}+\sqrt{49}=4+7=11$.

$\therefore\ x=1$ is a solution of the given equation ; but $x=16$ is not a solution if, as was agreed on p. 41, the symbol $\sqrt{x}$ means the *positive* square root of x.

The process of squaring each side of an equation may, as above, introduce extra roots ; *it is therefore essential to test the results, by substituting in the original equation.* The root $x=16$ is the solution of $\sqrt{(3x+1)}-\sqrt{x}=3$. If we solve this equation by the method given above, we shall find it leads to the same equation as before, $x^2-17x+16=0$, giving $x=1$ or $x=16$; but now we must reject $x=1$ and take $x=16$.

Before squaring each side, arrange the equation so that the process of squaring reduces the number of radical signs ; *e.g.* in $\sqrt{(x+1)}+x=11$, if we square each side as it stands we still have a radical sign left ; but if we write it in the form $\sqrt{(x+1)}=11-x$, the process of squaring removes the radical sign.

Example 9. Express with a rational denominator $\dfrac{1}{3+\sqrt{2}-\sqrt{5}}$.

$$\frac{1}{3+\sqrt{2}-\sqrt{5}}=\frac{3+\sqrt{2}+\sqrt{5}}{[(3+\sqrt{2})-\sqrt{5}][(3+\sqrt{2})+\sqrt{5}]}$$

$$=\frac{3+\sqrt{2}+\sqrt{5}}{(3+\sqrt{2})^2-(\sqrt{5})^2}=\frac{3+\sqrt{2}+\sqrt{5}}{9+2+6\sqrt{2}-5}$$

$$=\frac{3+\sqrt{2}+\sqrt{5}}{6+6\sqrt{2}}=\frac{3+\sqrt{2}+\sqrt{5}}{6(1+\sqrt{2})}$$

$$=\frac{(3+\sqrt{2}+\sqrt{5})(\sqrt{2}-1)}{6(\sqrt{2}+1)(\sqrt{2}-1)}$$

$$=\frac{3\sqrt{2}-3+2-\sqrt{2}+\sqrt{10}-\sqrt{5}}{6(2-1)}$$

$$=\frac{2\sqrt{2}+\sqrt{10}-\sqrt{5}-1}{6}$$

EXERCISE III. d

1. *Write down* the squares of

(i) $5-\sqrt{3}$; (ii) $3+\sqrt{7}$; (iii) $\sqrt{7}-\sqrt{3}$; (iv) $\sqrt{10}+\sqrt{6}$.

2. Find by inspection rational values of x and y such that:

(i) $x+y+2\sqrt{(xy)}=5+2\sqrt{6}$;

(ii) $x+y-2\sqrt{(xy)}=12-2\sqrt{35}$.

What are the square roots of $5+2\sqrt{6}$ and $12-2\sqrt{35}$?

3. Verify that $\sqrt{(3\frac{1}{2})}+\sqrt{(1\frac{1}{2})}$ is a square root of $5+\sqrt{21}$. *Write down* a square root of $5-\sqrt{21}$.

Find the positive square roots of the following:

4. $11+6\sqrt{2}$. 5. $7-4\sqrt{3}$. 6. $53+20\sqrt{7}$.

7. $5-2\sqrt{6}$. 8. $18+12\sqrt{2}$. 9. $15(4-\sqrt{15})$.

10. Simplify $\sqrt{(2+\sqrt{3})}-\sqrt{(2-\sqrt{3})}$.

11. Simplify $(2+\sqrt{3})^3-(2-\sqrt{3})^3$.

12. Can you find rational values of x and y such that $x+y\sqrt{2}$ equals $(3-\sqrt{2})^3$?

13. Simplify $\{1+\sqrt{(1-a)}\}\{1-\sqrt{(1-a)}\}$.

Solve the following equations:

14. $2x-\sqrt{x}=6$. 15. $5+\sqrt{(x-2)}=2x$.

16. $\sqrt{(3x+1)}=\frac{1}{2}(x+3)$. 17. $\sqrt{(8x)}+\sqrt{(2x)}=30$.

18. $\sqrt{(x+8)}-1=\sqrt{(3x+1)}$. 19. $\sqrt{(x+1)}=3-\sqrt{(x-2)}$.

20. $\sqrt{(12+x)}-\sqrt{(4+x)}=\sqrt{(1-x)}$.

21. $\sqrt{(3x+1)}+\sqrt{(2x+2)}=\sqrt{(2x+3)}+\sqrt{(3x)}$.

22. (i) Simplify $(\sqrt{2}+\sqrt{3}+\sqrt{6})(\sqrt{2}+\sqrt{3}-\sqrt{6})$.

(ii) Express with a rational denominator $\dfrac{\sqrt{2}}{\sqrt{2}+\sqrt{3}+\sqrt{6}}$.

Express with rational denominators Nos. 23-25:

23. $\dfrac{\sqrt{3}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$. 24. $\dfrac{1}{1+\sqrt{7}-\sqrt{5}}$. 25. $\dfrac{\sqrt{10}}{\sqrt{7}-\sqrt{5}-\sqrt{2}}$.

26. Simplify $\sqrt{\left\{\dfrac{\sqrt{7}-\sqrt{6}}{\sqrt{7}+\sqrt{6}}\right\}}$.

27. The following equations occur in the restricted theory of relativity:

$$x_1=\frac{x-ut}{\sqrt{(1-u^2)}} \quad\text{and}\quad t_1=\frac{t-ux}{\sqrt{(1-u^2)}}.$$

Prove that (i) $\dfrac{x_1+ut_1}{\sqrt{(1-u^2)}}=x$; (ii) $\dfrac{t_1+ux_1}{\sqrt{(1-u^2)}}=t$.

28. With the data of No. 27, prove that

$$t_1^2 - x_1^2 = t^2 - x^2.$$

[*For further practice, see Appendix, Ex. F.P.* 4, *Nos.* 23-32, *p.* 157.]

TEST PAPERS C. 1-10

[*For test papers on the work of Parts I-II, see Appendix, Y* 1-10, *p.* 180.]

C. 1

1. Find to 3 figures the values of

$$\text{(i) } \frac{1{\cdot}472 \times 1728}{92{\cdot}43}; \quad \text{(ii) } \sqrt[3]{(14{\cdot}97)}.$$

2. Express as powers of x,

$$\frac{x^5}{x^7}; \quad \frac{x}{x^2}; \quad \frac{x}{\sqrt{x}}; \quad (\sqrt{x})^3; \quad \frac{x^5 \times x^{10}}{x^{15}}.$$

3. The volume of a sphere, radius r inches, is $\frac{4}{3}\pi r^3$ cu. in. Find the volume of a sphere of radius 2·07 inches.

4. Simplify: (i) $3 + \dfrac{x^2+x}{x^2-3x+2} - \dfrac{6}{x-2}$;

(ii) $\dfrac{(4a^2-3a-10)(3a-6)}{3a^2-12}$.

5. If a body moves s feet in t seconds from rest, under the action of a constant force, then s varies as the square of t. If it moves 3 feet in the first 2 seconds, find how far it will move in (i) the first 4 seconds, (ii) the fifth second, (iii) the nth second.

C. 2

1. Find to 3 figures the values of

$$\text{(i) } (1{\cdot}172)^3 + (0{\cdot}9168)^3; \quad \text{(ii) } \sqrt[5]{(146{\cdot}8)}.$$

2. Evaluate (i) $27^{1\frac{1}{3}}$; (ii) $32^{0\cdot8}$; (iii) $8^{-\frac{2}{3}}$; (iv) $(0{\cdot}8)^{-1}$.

3. Show that the values of x, y in the given table obey the law $y \propto x^3$.

x . . .	2	3	5	10
y . . .	4·8	16·2	75	600

(i) What is y if $x=20$? (ii) What is x if $y=0{\cdot}075$?

(iii) Find to the nearest unit the percentage increase in y if x is increased by 20 per cent.

4. Express in their simplest surd forms,

$$\text{(i) } 6\sqrt{(\tfrac{1}{2})};\quad \text{(ii) } \frac{\sqrt{96}}{\sqrt{18}};\quad \text{(iii) } \frac{3+\sqrt{3}}{3-\sqrt{3}}.$$

5. The receipts from the sale of entrance tickets to a cricket match were £400. It is estimated that if the price of a ticket is reduced by sixpence, 2000 more people will pay for admission; the receipts will then be £50 more. What was the original price of a ticket?

C. 3

1. Find to 3 figures the values of

$$\text{(i) } \frac{94{\cdot}78 \times 0{\cdot}000137}{0{\cdot}08123};\quad \text{(ii) } (0{\cdot}8168)^{\frac{2}{3}}.$$

2. Express as powers of 8:

$$2;\quad 16;\quad \tfrac{1}{8};\quad 1;\quad 32;\quad \tfrac{1}{4};\quad \sqrt{2};\quad 2^p \times 4^q.$$

3. (i) What is the value of c if $x+1$ is a factor of $9x^2 - 4x - c$?

(ii) Solve $2x + 3y = 1,\quad 3x^2 + 5xy - y^2 = 1.$

4. Express in their simplest forms:

$$\text{(i) } \sqrt{12} \times \sqrt{27};\quad \text{(ii) } \frac{\sqrt{3}}{\sqrt{27}};\quad \text{(iii) } \frac{6}{\sqrt{18}};\quad \text{(iv) } \sqrt{2} - \sqrt{(\tfrac{1}{2})}.$$

5. The cost of manufacturing a motor-car is assumed to consist of a fixed sum together with an additional sum which varies inversely as the number of cars produced per day. When the daily output is 40 cars, the cost of each is £170; for 50 cars daily, it is £160. Find the cost of each when the daily output is 80 cars.

C. 4

1. (i) Find the value of $\pi r^2 h$ when $r = 0{\cdot}76$ and $h = 0{\cdot}142$.

(ii) Evaluate $100 \div (3{\cdot}004)^2$.

2. What are the values of

$$\text{(i) } 16^{\frac{3}{4}};\quad \text{(ii) } 4^{-\frac{1}{2}};\quad \text{(iii) } (\tfrac{1}{2})^{-1};\quad \text{(iv) } (-1)^{-2}\,?$$

3. Simplify

$$\text{(i) } \frac{(9-4x^2)(4-x^2+3x)}{(3+x-2x^2)(x-4)};$$

$$\text{(ii) } \frac{3}{y+2} - \frac{2}{y+3} + \frac{y+1}{y^2+5y+6}.$$

4. (i) Find a simple surd form for $\sqrt{(9-6\sqrt{2})}$.

(ii) Simplify $\sqrt{(9-6\sqrt{2})} \times \sqrt{(9+6\sqrt{2})}$.

5. The distance of the visible horizon at sea varies as the square root of the observer's height above sea level. If the distance is 3 miles for a height of 6 feet, find a formula for the height h feet at which an observer must be placed to see d miles.

C. 5

1. Find x if (i) $3^x = 81$; (ii) $81^x = 3$; (iii) $3^x = \frac{1}{9}$;
(iv) $9^{-x} = 3$; (v) $27^x = \sqrt{3}$; (vi) $27^{x+1} = 81^{2x}$.

2. Find l to three figures if $t = 2\pi\sqrt{\left(\frac{l}{g}\right)}$, when $t = 1{\cdot}20$; $g = 981$.

3. (i) Factorise $42a^2 + ab - 30b^2$; (ii) Solve $42x - \frac{30}{x} = 1$;
(iii) Solve $p(q-1) = 8$; $q(p-1) = 9$.

4. Evaluate $x^2 - \frac{1}{x^2}$ if $x = 3 - \sqrt{7}$.

5. In 4 years from now, A will be $\frac{2}{3}$ of B's present age; in 11 years from now, B will be double C's present age; 14 years ago, C was half of A's present age. Find the present ages of A, B, C.

C. 6

1. Evaluate (i) $(7{\cdot}22)^{\frac{2}{3}}$; (ii) $(0{\cdot}722)^{\frac{2}{3}}$.

2. Express as powers of y,

(i) $\frac{1}{\sqrt[3]{y}}$; (ii) $\sqrt[3]{y^2} \times \sqrt{y^3}$; (iii) $\frac{\sqrt{y} \times \sqrt{y}}{y}$; (iv) $\frac{y^{2n}}{y^{n+2}}$.

3. Solve the equations:

(i) $7x = 4 - 2x^2$; (ii) $\sqrt{(2y+1)} - \sqrt{y} = 1$.

4. Express in their simplest surd forms:

(i) $\sqrt{54} \times \sqrt{75}$; (ii) $\sqrt{32} \div \sqrt{12}$;

(iii) $\frac{6}{\sqrt{5} - \sqrt{2}}$; (iv) $(\sqrt{\tfrac{1}{2}})^3$

5. The height that the outer rail should be raised above the inner at a curve on a railway varies directly as the gauge of the railway and the square of the greatest train-speed allowed and inversely as the radius of the curve. In what ratio should this height be altered if the gauge is increased by 20 per cent., the speed by 50 per cent., and the radius by 100 per cent.?

C. 7

1. Find to 3 figures the value of t given by the equation, $\frac{1}{2}gt^2 = 90{\cdot}5$, when $g = 32{\cdot}2$.

2. (i) Express in prime factors $6^n \times 9^{n+1} \times 12^{n-2}$.

(ii) Simplify $\sqrt{75} + \sqrt{48} - 2\sqrt{27}$.

3. (i) Given $\sqrt{5} \simeq 2{\cdot}236$, find the value of x to 3 figures without using tables, if $x + x\sqrt{5} = 20$.

(ii) Solve $y^2 - 2y\sqrt{5} = 4$ (to 3 figures).

4. If a howitzer is fired at a given elevation, the range varies as the square of the muzzle velocity. If the range is 5000 yards when the muzzle velocity is 600 ft. per sec., what muzzle velocity would be required for a range of 6000 yards ?

5. The sum of the cubes of the whole numbers from 1 to n inclusive is $\frac{n^2(n+1)^2}{4}$. Use this fact to find the sum of the cubes of the whole numbers from 11 to 20 inclusive.

C. 8

1. Find to 3 figures the value of

$$\text{(i) } \left(\frac{417{\cdot}2 \times 0{\cdot}0468}{91{\cdot}48 \times 3712}\right)^{\frac{1}{3}}; \quad \text{(ii) } (0{\cdot}003)^{-\frac{1}{3}}.$$

2. Simplify (i) $\sqrt{6} \times \sqrt{10} \times \sqrt{20}$; (ii) $(\sqrt{10} - \sqrt{2})^2$.

3. Express in its simplest form, with a rational denominator,

$$\frac{7}{(3-\sqrt{2})(\sqrt{6}-\sqrt{3})}.$$

4. Solve the equations :

$$\text{(i) } W(x-2)=10, \quad W(2x-3)=27;$$
$$\text{(ii) } \sqrt{(x+7)} = 1 + \sqrt{(3x-2)}.$$

5. The safe load that can be carried by a beam of rectangular section varies directly as the breadth and the square of the depth and inversely as the length of the beam. What is the percentage change in the safe load if the breadth is halved, the depth increased by 50 per cent. and the length increased by 25 per cent. ?

How does the safe load vary with the length, breadth and volume of the beam ?

C. 9

1. Find to 3 figures the values of

$$\text{(i) } (4{\cdot}68)^{\frac{2}{3}}; \quad \text{(ii) } (0{\cdot}00468)^{\frac{3}{2}}; \quad \text{(iii) } (0{\cdot}1)^{0{\cdot}2}.$$

2. Simplify

$$\text{(i) } \frac{\sqrt{27} \times \sqrt{32}}{\sqrt{50}}; \quad \text{(ii) } \frac{1}{\sqrt{3}-\sqrt{2}} + \frac{2}{\sqrt{3}+\sqrt{2}};$$
$$\text{(iii) } \sqrt{(14+6\sqrt{5})}.$$

3. If $y = a + bx + cx^2$ where a, b, c are constant, and if the values of y when x is equal to $-\frac{1}{2}, \frac{1}{2}, 1\frac{1}{2}$ are $5\frac{1}{2}, 2\frac{1}{2}, -3\frac{1}{2}$ respectively, find the values of a, b, c.

4. The official H.P. rating of a car varies directly as the number of cylinders and the square of their diameter. A four-cylinder car with cylinders of diameter 3″ is rated as 14·4 H.P.; what is the H.P. of a six-cylinder car with cylinders of diameter 2·5″ ?

5. The denominator of a fraction is positive and is 3 more than its numerator. A new fraction is formed by adding 7 to both numerator and denominator. If the product of the two fractions is $\frac{1}{2}$, find the original fraction.

C. 10

1. Find to 3 figures the value of

$$\text{(i)}\ \sqrt[3]{\left(\frac{47{\cdot}18 \times 1{\cdot}473}{328{\cdot}7}\right)};\quad \text{(ii)}\ \frac{1}{(0{\cdot}2807)^3}.$$

2. Simplify (i) $\dfrac{12}{\sqrt{5}-\sqrt{2}}$; (ii) $\sqrt{3}-\sqrt{(\frac{1}{3})}$; (iii) $(1{\cdot}5)^{2n} \times 12^n$.

3. (i) Simplify $\dfrac{1}{y+3}-\dfrac{2y}{y^2-9}$;

(ii) If $y=\dfrac{3-2z}{4z-8}$ and if $z=\dfrac{5+4x}{3x+2}$, find x in terms of y.

4. The expense of a ship's voyage between two places is the sum of two parts, one of which varies directly, and the other of which varies inversely, as the number of days the voyage lasts. If the time taken is 10 days, the expense is £9200, and for 14 days it is £10,000. What is it for 12 days?

5. An express, travelling at a certain rate, is 2 minutes late when passing A, and 4 minutes late when passing B, 40 miles further on. Its speed is then increased by 12 miles an hour so that it passes C, 20 miles further on, punctually. At what rate in miles per hour would it have travelled if it had been running to time all the way?

[*For additional test papers, see Appendix, Z.* 1-10, *p.* 184.]

CHAPTER IV

THEORY OF LOGARITHMS AND HARDER INDICES

THE general logarithm is defined as follows :

If $x = a^m$, then m is called the logarithm of x to base a, and we write

$$\mathbf{m = \log_a x.}$$

Thus the logarithm of a number x to any base a is the power to which the base must be raised to give the number ; in short, a logarithm is always an index.

The definition may be expressed also in the form :

$$\mathbf{\log_a (a^m) = m} ;$$

and in particular, $\log_a 1 = 0$, since $a^0 = 1$;

and $\qquad \log_a a = 1$, since $a = a^1$.

Properties of General Logarithms.

$$\text{(i) } \mathbf{\log_a (xy) = \log_a x + \log_a y.}$$

Let $\log_a x = m$ and $\log_a y = n$.

Then $\qquad x = a^m \quad \text{and} \quad y = a^n$;

$$\therefore\ xy = a^m \times a^n = a^{m+n} ;$$

$$\therefore\ \log_a xy = m + n = \log_a x + \log_a y.$$

$$\text{(ii) } \mathbf{\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y.}$$

With the same notation as before,

$$\frac{x}{y} = a^m \div a^n = a^{m-n} ;$$

$$\therefore\ \log_a \left(\frac{x}{y}\right) = m - n = \log_a x - \log_a y.$$

$$\text{(iii) } \mathbf{\log_a (x^p) = p \log_a x.}$$

Let $\log_a x = m$, $\quad \therefore\ x = a^m$;

$$\therefore\ x^p = (a^m)^p = a^{mp} ;$$

$$\therefore\ \log_a (x^p) = mp = p \times m = p \log_a x.$$

From this result, we have the following:

$$\log_a\left(\frac{1}{x^q}\right)=\log_a(x^{-q})=-q\log_a x;$$

and $$\log_a(\sqrt[n]{x})=\log_a\left(x^{\frac{1}{n}}\right)=\frac{1}{n}\log_a x.$$

If we have a table of logarithms calculated for some given base a, we can deduce the logarithms for any other base b by the following relation:

$$\text{(iv) } \log_b \mathbf{x}=\frac{\log_a \mathbf{x}}{\log_a \mathbf{b}}.$$

Let $\log_b x=m$, $\therefore\ x=b^m$;

$$\therefore\ \log_a x=\log_a(b^m)=m\log_a b;$$

$$\therefore\ \frac{\log_a x}{\log_a b}=m=\log_b x.$$

In particular, if in this result we put $x=a$, we have

$$\log_b a=\frac{\log_a a}{\log_a b}=\frac{1}{\log_a b}.$$

Example 1. What is the value of $\log_8 32$?

If $\log_8 32=n$, $32=8^n$.

But $\quad 32=2^5$ and $8=2^3$, $\quad\therefore\ 2^5=(2^3)^n=2^{3n}$;

$$\therefore\ 3n=5;\quad \therefore\ n=\frac{5}{3}.$$

Or, $\quad 32=2^5=(2^3)^{\frac{5}{3}}=8^{\frac{5}{3}}$; $\quad\therefore\ \log_8 32=\frac{5}{3}$.

Or, $$\log_8 32=\frac{\log_2 32}{\log_2 8}=\frac{\log_2(2^5)}{\log_2(2^3)}=\frac{5}{3}.$$

Example 2. Simplify $\dfrac{\log 125}{\log 25}$.

$$\log 125=\log(5^3)=3\log 5;$$

$$\log 25=\log(5^2)=2\log 5.$$

$$\therefore\ \text{expression}=\frac{3\log 5}{2\log 5}=\frac{3}{2}.$$

Note. If the base is not specified, as here, it is implied that all logarithms which occur have the same base, unless otherwise stated. Thus the expression in Example 2 is an abbreviation for $\dfrac{\log_a 125}{\log_a 25}$.

EXERCISE IV. a

Express the following in the form $\log x$:

1. $\log 2 + \log 5$. 2. $\log 12 - \log 3$. 3. $3 \log 2$.

4. $-\log 4$. 5. $2 \log 5 + \log 1$. 6. $\frac{1}{2} \log 16$.

Write down the values of

7. $\log_{10} 100$. 8. $\log_{10} (0{\cdot}1)$. 9. $\log_2 8$. 10. $\log_2 2$.

11. $\log_4 16$. 12. $\log_4 \frac{1}{4}$. 13. $\log_3 81$. 14. $\log_3 \frac{1}{3}$.

15. $\log_9 1$. 16. $\log_9 3$. 17. $\log_9 27$. 18. $\log_{\frac{1}{2}} 8$.

Write down the values of x for the following equations:

19. $\log_{10} x = 2$. 20. $\log_{10} x = \frac{1}{2}$. 21. $\log_5 x = 3$.

22. $\log_4 x = 1$. 23. $\log_4 x = 0$. 24. $\log_4 x = \frac{1}{4}$.

25. $\log_9 x = -1$. 26. $\log_9 x = \frac{1}{2}$. 27. $\log_9 x = -\frac{1}{2}$.

28. $\log_9 x = \frac{3}{2}$. 29. $\log_8 x = \frac{2}{3}$. 30. $\log_8 x = -\frac{1}{3}$.

Express the following in the form $\log_a x$:

31. $1 + \log_{10} 3$. 32. $2 + \log_3 5$. 33. $\frac{1}{2} + \log_9 7$.

34. $2 - \log_5 4$. 35. $\frac{1}{3} - 2 \log_3 5$. 36. $\frac{1}{4} \log 4 + \frac{1}{2} \log 2$.

37. Given $\log_{10} 2 \simeq 0{\cdot}301$ and $\log_{10} 3 \simeq 0{\cdot}477$, obtain the values of $\log_{10} x$ for $x = 20, 18, 16, 15, 5, 0{\cdot}5, 0{\cdot}2$.

38. Find the value of x correct to 3 significant figures, if $10^x = \sqrt[3]{2}$.

39. Write down the logarithms to base 2 of

(i) 2^n; (ii) $\frac{1}{4}$; (iii) $\sqrt[3]{32}$; (iv) $8 \times 16^{n-2}$.

40. Write down the logarithms to base 8 of

(i) 2; (ii) 4; (iii) 16; (iv) $0{\cdot}5$; (v) $0{\cdot}125$.

Simplify the following:

41. $\dfrac{\log 8}{\log 2}$. 42. $\dfrac{\log 4}{\log \frac{1}{4}}$. 43. $\dfrac{\log 27}{\log 9}$.

44. $\log x^2 - \log xy$. 45. $\log z + \log \dfrac{1}{z}$.

46. $\dfrac{\log x^3 - \log x}{\log x^2 - \log x}$. 47. $\log(\log y^2) - \log(\log y)$.

48. $\log_{10} 125 + \log_{10} 32 - \log_{10} 4$. 49. $\log_{10}(\sqrt{10} \times \sqrt[3]{10})$.

50. $\log_{10} 75 + \log_{10} 28 - \log_{10} 0{\cdot}21$. 51. $10^{\log_{10} 2}$.

52. Use tables to evaluate $\log_3 4$ to 3 significant figures.

53. Use tables to evaluate $\log_e 10$, if $e = 2{\cdot}718$.

54. If $a = \log x$ and $b = \log y$ and $a + nb = \log z$, express z in terms of x, y, n.

[*For additional examples, see Appendix, Ex. T. 3, p.* 169.]

Compound Interest. The amount to which any sum of money £P accumulates at compound interest, r per cent. per annum, in n years may be calculated by logarithms. But if the number of years is large, four-figure tables do not give a sufficient degree of accuracy for practical purposes.

The interest for the first year is $£\mathsf{P} \times \frac{r}{100}$;

$\therefore$ the amount at the end of the first year is

$$£\left(\mathsf{P} + \frac{\mathsf{P}r}{100}\right) = £\mathsf{P}\left(1 + \frac{r}{100}\right);$$

$\therefore$ the amount at the end of the second year is

$$£\mathsf{P}\left(1 + \frac{r}{100}\right) \times \left(1 + \frac{r}{100}\right) = £\mathsf{P}\left(1 + \frac{r}{100}\right)^2.$$

Similarly, the amount at the end of the third year is

$$£\mathsf{P}\left(1 + \frac{r}{100}\right)^3;$$

and so on.

$\therefore$ the amount after n years is $£\mathsf{P}\left(1 + \frac{r}{100}\right)^n$.

Example 3. Find the amount of £160 at $4\frac{1}{2}$ per cent. per annum compound interest for 7 years.

The amount $£\mathsf{A} = £160\left(1 + \frac{4{\cdot}5}{100}\right)^7 = £160(1 + 0{\cdot}045)^7$;

$$\therefore\ \mathsf{A} = 160(1{\cdot}045)^7.$$

Take logarithms (to base 10) of each side.

$$\begin{aligned} \therefore\ \log \mathsf{A} &= \log 160 + \log(1{\cdot}045)^7 \\ &= \log 160 + 7 \log 1{\cdot}045 \\ &= 2{\cdot}2041 + 7 \times 0{\cdot}0191 \\ &= 2{\cdot}3378. \end{aligned}$$

$$\begin{array}{r} 2{\cdot}2041 \\ {\cdot}1337 \\ \hline 2{\cdot}3378 \end{array}$$

$$\therefore\ \mathsf{A} = 217{\cdot}6.$$

$\therefore$ the amount is £218 correct to the nearest £.

Although $\log 1{\cdot}045$ equals $0{\cdot}0191$ correct to 4 places of decimals, the product $7 \times 0{\cdot}0191$ is not necessarily correct to 4 places of decimals. We cannot therefore expect in this example to obtain a higher degree of accuracy than the nearest £ in the answer.

Example 4. After how many years will £160 amount to £500 t $3\frac{1}{2}$ per cent. compound interest?

Suppose the number of years is n.

Then $$160\left(1+\frac{3\cdot5}{100}\right)^n=500;$$
$$\therefore\ (1\cdot035)^n=\tfrac{500}{160}.$$

Take logarithms (to base 10) of each side:

$$\therefore\ n\log 1\cdot035=\log 500-\log 160;$$
$$\therefore\ n\times0\cdot0149=2\cdot6990-2\cdot2041=0\cdot4949;$$
$$\therefore\ n=\frac{0\cdot4949}{0\cdot0149}.$$

The value of n may now be found by long division, or we may gain take logarithms, as follows:

$$\begin{aligned}\log n&=\log 0\cdot4949-\log 0\cdot0149\\&=\bar{1}\cdot6945-\bar{2}\cdot1732\\&=1\cdot5213.\\ \therefore\ n&=33\cdot2;\end{aligned}$$

$$\begin{array}{r}\bar{1}\cdot6945\\ \bar{2}\cdot1732\\ \hline 1\cdot5213\end{array}$$

$\therefore$ the length of time is 33 years, correct to the nearest year.

Note. Since the denominator $0\cdot0149$ in the expression for n is iven only to 3 significant figures, it is absurd to give more than significant figures in the answer, and we cannot even be sure hat the third figure in $33\cdot2$ is correct.

Example 5. If $2\log_{10}y+3\log_{10}x=0\cdot7$, express y in terms of x.

$$\log_{10}y+1\cdot5\log_{10}x=0\cdot35;$$

ut, from the tables, $10^{0\cdot35}\simeq2\cdot24$;

$$\therefore\ \log_{10}y+\log_{10}x^{1\cdot5}=\log_{10}2\cdot24;$$
$$\therefore\ \log(yx^{1\cdot5})=\log 2\cdot24;\quad \therefore\ yx^{1\cdot5}=2\cdot24;$$
$$\therefore\ \mathbf{y}=2\cdot24\mathbf{x}^{-1\cdot5}.$$

EXERCISE IV. b

[*In this Exercise,* $\log x$ *means* $\log_{10}x$.]

Write in a form not involving logarithmic notation the follow-ng equations:

1. $\log x+\log y=\log 2$.
2. $\log x+\log y=2$.
3. $\log x-\log y=3$.
4. $2\log x=3\log y$.
5. $3\log x-4\log y=1$.
6. $x\log 5=\log 2$.
7. $x\log 3=y\log 4$.
8. $\log y=x\log 3+\log 4$.
9. $\log y=1\cdot6\log x+0\cdot58$.
10. $\log y=0\cdot17x+1\cdot46$.

Solve, as accurately as the tables permit, the following:

11. $2^x = 10$. 12. $3^x = 100$. 13. $5^x = \sqrt{10}$.

14. $7^x = 14$. 15. $(6\cdot23)^x = 52$. 16. $(0\cdot06)^x = 0\cdot15$.

17. Find, as accurately as the tables permit, the amount at compound interest of

(i) £240 for 9 years at 4 per cent.;

(ii) £175 for 20 years at $2\frac{1}{2}$ per cent. (See p. 76.)

18. Find, as accurately as the tables permit, the sum of money which amounts at compound interest to £350 in 12 years at $3\frac{1}{2}$ per cent.

19. After how many years will a sum of money be doubled, if invested at compound interest (i) at 5 per cent.; (ii) at $5\frac{1}{2}$ per cent.?

20. Find the least integral value of n for which $(0\cdot95)^n$ is less than $0\cdot1$.

21. Find x if $3^{\log 2} = 2^x$.

22. If $pv^n = 13800$, and if $p = 7\cdot62$ and $v = 82\cdot3$, find the value of n.

23. Find the value of x correct to 2 significant figures, if $7^{x+1} = 4^{2x-3}$.

24. If $r = \frac{1}{3}$, find the least integral value of n for which $\dfrac{r^n}{1-r}$ is less than $0\cdot00001$.

25. If $y = ax^n$, where a and n are constants, and if for $x = 2$, $y = 10\cdot6$ and for $x = 3$, $y = 6\cdot2$, find the values of a and n.

Harder Examples on Indices

Example 6. Simplify (i) $(9x^{-6})^{-\frac{3}{2}}$; (ii) $\dfrac{18\,.\,12^{3n}}{8^{2n+1}\,.\,9^n}$.

(i) $$(9x^{-6})^{-\frac{3}{2}} = 9^{-\frac{3}{2}}x^9$$

$$= \frac{x^9}{9^{\frac{3}{2}}} = \frac{x^9}{(\sqrt{9})^3} = \frac{x^9}{27}.$$

(ii) $$\frac{18\,.\,12^{3n}}{8^{2n+1}\,.\,9^n} = \frac{2\,.\,3^2\,.\,(2^2\,.\,3)^{3n}}{(2^3)^{2n+1}\,.\,(3^2)^n}$$

$$= \frac{2\,.\,3^2\,.\,2^{6n}\,.\,3^{3n}}{2^{6n+3}\,.\,3^{2n}} = \frac{2^{6n+1}\,.\,3^{3n+2}}{2^{6n+3}\,.\,3^{2n}}$$

$$= \frac{3^{n+2}}{2^2} = \tfrac{1}{4}\,.\,3^{n+2}.$$

Example 7. Expand $(x^{\frac12}+\sqrt2+3x^{-\frac12})(x^{\frac12}-\sqrt2+3x^{-\frac12})$.

By ordinary multiplication, the expression equals

$$x^{\frac12}(x^{\frac12}-\sqrt2+3x^{-\frac12})+\sqrt2(x^{\frac12}-\sqrt2+3x^{-\frac12})+3x^{-\frac12}(x^{\frac12}-\sqrt2+3x^{-\frac12})$$

$$=x-\sqrt2x^{\frac12}+3+\sqrt2x^{\frac12}-2+3\sqrt2x^{-\frac12}+3-3\sqrt2x^{-\frac12}+9x^{-1},$$

since $x^0=1$,

$$=x+4+\frac9x, \text{ since } x^{-1}=\frac1x.$$

Or, Expression $=\{(x^{\frac12}+3x^{-\frac12})+\sqrt2\}\{(x^{\frac12}+3x^{-\frac12})-\sqrt2\}$

$$=(x^{\frac12}+3x^{-\frac12})^2-(\sqrt2)^2=x+6+9x^{-1}-2$$

$$=x+4+\frac9x, \text{ as before.}$$

EXERCISE IV. c

Simplify and express with positive indices :

1. $(2\frac12a)^{-1}$.
2. $(2\frac14b^3)^{\frac12}$.
3. $\left(\frac{8a^3}{b^2}\right)^{-\frac13}$.
4. $(x^{\frac12}y^{-\frac12})^{-2}$.
5. $\sqrt{\left(\frac{y}{z}\right)}\div(y^{\frac23}z^{-\frac16})^{\frac34}$.
6. $\frac{(16x^6y^{-2})^{-\frac14}}{x^{\frac12}y^{-\frac32}}$.

Simplify the following :

7. $27^{-\frac23}$.
8. $10^{1\cdot7}\times10^{1\cdot3}$.
9. $10^{2\cdot1}\times\sqrt{(10^{1\cdot8})}$.
10. $(1\frac{25}{144})^{-\frac32}$.
11. $\sqrt2\times\sqrt[3]{2}\times\sqrt[6]{2}$.
12. $4^{2n}\div2^n$.
13. $\frac{2^{n+1}-2^{n-1}}{2^n}$.
14. $2^m \,.\, 3^n \,.\, 6^{m-n}$.
15. $12^n\div2^{2n}$.
16. $\frac{5\,.\,4^{3n+1}}{20\,.\,8^{2n}}$.
17. $\frac{4^m\,.\,27^{m-n}}{6^{2m}}$.
18. $\frac{4^{n+1}\,.\,8^{1-n}}{16^{3-n}}$.
19. Solve (i) $9^x=27$; (ii) $27^x=9$.
20. If $x=\sqrt2-1$, prove that $x^2+x^{-2}=6$.
21. If $m=a^x$ and $n=a^y$ and $m^y\,.\,n^x=a$, find y in terms of x.
22. If $a=x^p$, $b=x^q$, $c=x^r$, express $\sqrt{\left(\frac{ab}{c^3}\right)}$ as a power of x.
23. If $t=8p\,.\,v^{1\cdot5}$, express v in terms of p, t.
24. Simplify $9^{2n+1}\times6^{2n-3}\div(18\times4^{n-2})$.
25. Expand $(x^{\frac12}+x^{-\frac12})(x-1+x^{-1})$.
26. Simplify $\frac{x^{3a}-y^{3b}}{x^{2a}-y^{2b}}-x^a-y^b$.

27. If $x^{\frac{3}{4}}y^{\frac{1}{3}}z^{-\frac{1}{2}} = 2$, express x in the form $ky^m z^n$, giving the value of k correct to 3 significant figures.

28. If $x^{-\frac{1}{2}}y^{\frac{2}{3}}z^{-2} = 3$ and if $x = y = 3$, express z as a power of 3.

29. Simplify $(a^2b^{-1})^{-2} \times (a^{-4}b^3)^{-1}$.

30. Multiply $2x - x^{\frac{1}{2}} + x^{-\frac{1}{2}}$ by $3x^{\frac{1}{2}} - 2$.

31. Simplify $(x - y) \div (x^{\frac{1}{2}} - y^{\frac{1}{2}})$.

32. If $\mathsf{H} = 0{\cdot}115\mathsf{Q}^3\mathsf{D}^{-4}$ express D in the form $k\mathsf{H}^m\mathsf{Q}^n$, giving the value of k correct to 3 significant figures.

33. Simplify the relation, $\sqrt{x} + \sqrt{y} = \sqrt{(x + y)}$.

34. Subtract $\dfrac{a^n}{(a+b)^m}$ from $\dfrac{a^{n-1}}{(a+b)^{m-1}}$.

35. What is the square root of $x - 2 + x^{-1}$?

36. Find values of x and y such that

$$2^{x+y} = 8\,; \quad 3^{x-y} = 1.$$

37. Solve the equation, $25^x = 5^{x+1} - 6$.

38. Multiply $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.

39. Make g the subject of the formula, $d = 1{\cdot}25g^{\frac{3}{4}} \,.\, h^{-\frac{1}{4}}$.

40. Simplify and express with radical signs,

$$(a + 2b)^{\frac{1}{2}}(a - 2b)^{-\frac{1}{3}}(a^2 - 4b^2)^{-\frac{1}{6}}.$$

[*For additional examples, see Appendix, Ex. T.* 4, *p.* 170. *For a revision exercise on Ch. III-IV, see Appendix, Ex. W.* 2, *p.* 147.]

CHAPTER V

SERIES

If we write down in succession a set of numbers in accordance with some law, the collection of numbers is called a **series** and each number in the collection is called a **term** of the series.

Example 1. What are the first 4 terms of the series whose nth term is $2n^2+1$ and what is the 20th term ?

The 1st term is $2 \,.\, 1^2+1=2+1=3$;
The 2nd term is $2 \,.\, 2^2+1=8+1=9$;
The 3rd term is $2 \,.\, 3^2+1=18+1=19$;
The 4th term is $2 \,.\, 4^2+1=32+1=33$.
The 20th term is $2 \,.\, 20^2+1=800+1=801$.

The series is therefore the set of numbers,

$$3,\ 9,\ 19,\ 33,\ \dots,\ (2n^2+1),\ \dots\,.$$

The nth term of a series is often called the **general term**, because if its form is given, any term of the series can be computed.

Example 2. Find a law which gives the following series :

$$4,\ 7,\ 10,\ 13,\ 16,\ 19,\ \dots\,.$$

One rule for this set of numbers may be stated as follows: Start from the number 4, and count upwards by threes ; thus, $4,\ 4+3,\ 4+3+3,\ 4+3+3+3,$ etc.

But the statement of the law is more compact and more useful, if we express the nth term as a function of n.

For this series, we find the 6th term by adding 5 threes to 4, and we find the 7th term by adding 6 threes to 4, and so on. We therefore find the nth term by adding $(n-1)$ threes to 4.

$$\therefore \text{ the } n\text{th term} = 4+(n-1)\times 3 = 4+3n-3$$
$$= 3n+1.$$

EXERCISE V. a

Write down the first four terms and the 10th term of the following series, Nos. 1-12 :

1. nth term $=3n-1$.   2. nth term $=3n+7$.

3. nth term $=10n^2$. 4. nth term $=\dfrac{1}{n+1}$.

5. nth term $=5\times 2^{n-1}$. 6. nth term $=(-1)^{n-1}$.

7. nth term $=11-5n$. 8. nth term $=13-5n$.

9. Start at 5 and count up by fours.

10. Start at 5 and count down by fours.

11. Odd numbers upwards in order, starting from 15.

12. Starting from 32, make each term half the preceding term.

Write down, in terms of n, the simplest form of the nth term of each of the following series, and check by using this form to compute the 4th term.

13. 3, 4, 5, 6, 7, ... 14. 2, 4, 6, 8, 10, ...

15. (i) 1, 3, 5, 7, 9, ...; (ii) 5, 7, 9, 11, 13, ...

16. (i) 1, 4, 9, 16, 25, ...; (ii) 25, 36, 49, 64, 81, ...

17. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$. 18. $1\times 3,\ 3\times 5,\ 5\times 7,\ 7\times 9, \ldots$.

19. 3, 6, 12, 24, 48, ... 20. 7, 70, 700, 7000, ...

21. 11, 7, 3, -1, -5, ... 22. $5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}, \ldots$.

23. $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$. 24. $\frac{2}{3}, \frac{4}{9}, \frac{6}{27}, \frac{8}{81}, \ldots$.

25. The nth term of a series is $7n-25$.

(i) How many terms of the series are negative ?

(ii) Is 66 a term of the series ? If so, which term ?

(iii) Express the law of the series in words : start at ... and count ... .

26. The nth term of a series is $5\times 3^{n-1}$.

(i) How many terms of the series are less than 200 ?

(ii) Express the law of the series in words. (Cf. Nos. 9-12.)

27. The nth term of a series is $19-4n$.

(i) How many terms of the series are positive ?

(ii) Is -49 a term of the series ? If so, which term ?

(iii) Express the law of the series in words.

28. The nth term of a series is $30\times(\frac{1}{3})^{n-1}$.

(i) How many terms of the series are greater than 1 ?

(ii) Express the law of the series in words.

29. The nth term of a series is $9n+7$. What is the result of subtracting the $(k-1)$th term from the kth term ?

30. The nth term of a series is $7\times 8^{n-1}$. What is the result of dividing the kth term by the $(k-1)$th term ?

Arithmetical Progression

If a series is formed by starting with any number and counting upwards or downwards by some fixed amount, it is called an **arithmetical progression**, or, more shortly, an **A.P.**; and the number obtained by subtracting any term from the term which follows it is called the **common difference** of the A.P.;

e.g. 7, 11, 15, 19, 23, ... is an A.P., common difference, 4,
and 7, 3, -1, -5, -9, ... is an A.P., common difference, -4;
and $2\frac{1}{4}$, $1\frac{1}{2}$, $\frac{3}{4}$, 0, $-\frac{3}{4}$, $-1\frac{1}{2}$, ... is an A.P., common difference, $-\frac{3}{4}$.

Example 3. A man starts with a salary of £250 a year and receives annual increases of £20 a year. How much does he receive for the nth year of service?

His salary for successive years is as follows:

£250, £270, £290, £310, ...

The set of numbers 250, 270, 290, 310, ... form an A.P.

The 1st term $= 250$.
The 2nd term $= 250 + 20$.
The 3rd term $= 250 + 2 \times 20$.
The 4th term $= 250 + 3 \times 20$; and so on.
The nth term $= 250 + (n-1) \times 20$
$= 250 + 20n - 20 = 230 + 20n$.

$\therefore$ for the nth year, he receives **£$(230 + 20n)$**.

Arithmetic Mean. If we take any 3 numbers in A.P., the middle term is called the **Arithmetic Mean** of the two outside terms. Thus since 11, 18, 25 are in A.P., the Arithmetic Mean of 11 and 25 is 18; it is simply the *average* of the two numbers.

If x, y, z are in A.P., the common difference is $y - x$ and also $z - y$; $\therefore y - x = z - y$;

$$\therefore 2y = z + x; \quad \therefore y = \tfrac{1}{2}(z + x).$$

$\therefore$ **the arithmetic mean of x, z is $\frac{1}{2}(x + z)$.**

EXERCISE V. b

1. A man, after obtaining a certain post, saves £20 his first year, £26 his second year, £32 his third year, and continues to increase his annual savings by £6 each year. What does he save in the nth year?

2. A swimming bath has a plane sloping floor; the depth of water is indicated by posts at equal distances down the bath. The reading on the first post is 12 ft., on the second post is

11 ft. 6 in. What is the reading on the 5th post ? on the nth post ?

How many posts are there altogether if the reading on the last post is 4 ft. ?

3. A marble rolls down an inclined groove; the distances it travels in successive seconds are 3 cm., 9 cm., 15 cm., 21 cm., etc. How far does it travel in the nth second ?

4. The temperature of the water in a boiler is rising at a steady rate; readings taken every 20 minutes are as follows: 82° F., 88° F., 94° F., 100° F., etc. The last reading was 190° F.; how many readings were taken in all ? What was the nth reading ?

Find the nth term in each of the following series in A.P.:

5. 9, 11, 13, 15, ...

6. 40, 37, 34, 31, ...

7. $2\frac{1}{2}$, $3\frac{1}{4}$, 4, $4\frac{3}{4}$, ...

8. 13, 7, 1, -5, ...

9. $\frac{a}{2}, a, \frac{3a}{2}, 2a, \ldots$.

10. $a, a+d, a+2d, a+3d, \ldots$.

Find the number of terms in the following series in A.P.:

11. 4, 9, 14, ..., 64.

12. 8, 5, 2, ..., -25.

13. 6, 5·9, 5·8, ..., 3·5.

14. 22, 29, 36, ..., 169.

Find the common difference and the second term in the following series in A.P.:

15. 1st term 5; 4th term 14.

16. 1st term 4; 3rd term 11.

17. 1st term 8; 10th term 71.

18. 1st term 7; 5th term 17.

19. 1st term 32; 6th term 12.

20. 1st term 10; 8th term -11.

21. nth term, $7n-3$.

22. nth term, $30-7n$.

23. What is the arithmetic mean of (i) 5, 13; (ii) 3, -7; (iii) $a+h$, $a-h$?

24. Write down the numbers 20 and 40, then insert between them three numbers so as to give five numbers in A.P.

This process is called *inserting* 3 *Arithmetic Means between* 20 and 40.

25. Insert 5 Arithmetic Means between 15 and 60.

26. Find the least number above 100 which belongs to the following A.P.:

(i) 6, 13, 20, 27, ... ; (ii) 1·3, 1·6, 1·9, 2·2, ...

27. The last term of an A.P. containing n terms is l; the common difference is d. What is the first term ?

28. The first and last terms of an A.P. are a, l, respectively. If there are n terms, what is the common difference ?

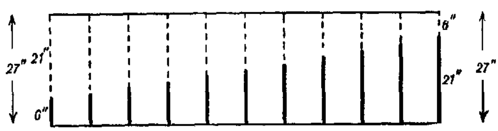
Sum of Numbers in A.P.

Example 4. An instrument for a jazz band is composed of a set of 11 thin metal tubes ; their lengths in inches are as follows :

$$6,\ 7\tfrac{1}{2},\ 9,\ 10\tfrac{1}{2},\ 12,\ 13\tfrac{1}{2},\ 15,\ 16\tfrac{1}{2},\ 18,\ 19\tfrac{1}{2},\ 21.$$

What is the total length of metal tubing required for the set ?

If there are only a few numbers, the quickest and simplest method is to add up in the ordinary way. But if the number of terms is large and if, as here, they form an A.P., it is easier to use a different method.



Set of 11 tubes

FIG. 4.

Fig. 4 represents a set of 11 tubes ranging from 6 inches to 21 inches. Suppose sets are manufactured as follows :

Take 11 tubes, each 27 inches long, and cut off in succession from them portions of lengths, as required :

$$6,\ 7\tfrac{1}{2},\ 9,\ 10\tfrac{1}{2},\ 12,\ 13\tfrac{1}{2},\ 15,\ 16\tfrac{1}{2},\ 18,\ 19\tfrac{1}{2},\ 21, \text{ inches.}$$

Then there remain tubes whose lengths in order are :

$$21,\ 19\tfrac{1}{2},\ 18,\ 16\tfrac{1}{2},\ 15,\ 13\tfrac{1}{2},\ 12,\ 10\tfrac{1}{2},\ 9,\ 7\tfrac{1}{2},\ 6, \text{ inches ;}$$

these make up exactly a second set.

But the total length of 11 tubes, each 27 inches long, is 11×27 inches, and these tubes form exactly 2 sets.

the length of tubing for each set is $\dfrac{11 \times 27}{2}$ **inches.**

The argument used in Example 4 may also be expressed in each of the following ways :

Example 5. What is the average of the 11 numbers :

$$6,\ 7\tfrac{1}{2},\ 9,\ 10\tfrac{1}{2},\ 12,\ 13\tfrac{1}{2},\ 15,\ 16\tfrac{1}{2},\ 18,\ 19\tfrac{1}{2},\ 21\ ?$$

Hence find their sum.

These numbers form an A.P. ; reading forwards from the left they increase by $1\tfrac{1}{2}$; reading backwards from the right, they decrease by $1\tfrac{1}{2}$.

$$6 + 21 = 7\tfrac{1}{2} + 19\tfrac{1}{2} = 9 + 18 = 10\tfrac{1}{2} + 16\tfrac{1}{2} = 12 + 15.$$

$\therefore$ the average of each of these pairs is $\frac{27}{2} = 13\frac{1}{2}$; there is also a middle term, $13\frac{1}{2}$.

$$\therefore \text{ the average of the whole set is } \tfrac{27}{2}.$$

$$\therefore \text{ the sum} = \frac{11 \times 27}{2}.$$

Example 6. What is the sum of the 11 numbers :

$$6,\ 7\tfrac{1}{2},\ 9,\ 10\tfrac{1}{2},\ 12,\ 13\tfrac{1}{2},\ 15,\ 16\tfrac{1}{2},\ 18,\ 19\tfrac{1}{2},\ 21\ ?$$

Let the sum of the numbers be S.

Then $S = 6 + 7\frac{1}{2} + 9 + 10\frac{1}{2} + 12 + 13\frac{1}{2} + 15 + 16\frac{1}{2} + 18 + 19\frac{1}{2} + 21$;

also $S = 21 + 19\frac{1}{2} + 18 + 16\frac{1}{2} + 15 + 13\frac{1}{2} + 12 + 10\frac{1}{2} + 9 + 7\frac{1}{2} + 6$;

$\therefore$ by addition,

$$\begin{aligned} 2S &= 27 + 27 + 27 + \dots\dots + 27 \\ &= 11 \times 27; \end{aligned}$$

$$\therefore S = \frac{11 \times 27}{2}.$$

EXERCISE V. c

Find the sum of the following series in A.P. :

1. 5, 7, 9, 11, ..., 21. (9 terms.)
2. 8, 11, 14, 17, ..., 65. (20 terms.)
3. 100, 93, 86, 79, ..., 23. (12 terms.)
4. 15, 11, 7, 3, ..., -45. (16 terms.)
5. 6, 7, 8, 9, ..., to 20 terms.
6. 20, 17, 14, 11, ..., to 15 terms.
7. 5, 5·2, 5·4, 5·6, ..., to 36 terms.
8. 1, 2, 3, 4, 5, ..., to n terms.
9. 1, 3, 5, 7, ..., to n terms.

10. The sum of the A.P., $3 + \dots + 59$, is 465. Find the number of terms and the common difference.

11. The sum of 12 terms of the A.P., $4 + \dots$, is 246. Find the last term and the common difference.

12. The sum of 20 terms of an A.P. is 1280 and the last term is 121. Find the first term and the common difference.

General Method

Suppose the first term of an A.P. is a and that the common difference is d, then the terms of the series are

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots;$$

$$\therefore \text{ the } n\text{th term is } a+(n-1)d.$$

Suppose there are n terms and that the last term is l.

Then $$l=a+(n-1)d.$$

Sum of an A.P. Let the sum of n terms be S.

If the last term is l, the last term but one is $l-d$, and the last term but two is $l-2d$, and so on.

$$\therefore\ S=a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l.$$

Now write this expression for S backwards.

$$\therefore\ S=l+(l-d)+(l-2d)+\ldots+(a+2d)+(a+d)+a.$$

$\therefore$ adding

$$2S=(a+l)+(a+l)+(a+l)+\ldots+(a+l)+(a+l).$$

But there are n brackets,

$$\therefore\ 2S=n(a+l);$$

$$\therefore\ S=\frac{n}{2}(a+l).$$

Further, $l=a+(n-1)d$, $\quad\therefore\ a+l=2a+(n-1)d$;

$$\therefore\ S=\frac{n}{2}[2a+(n-1)d].$$

This argument merely sets out in full that the average of a set of numbers in A.P. is found by taking the average of the first and last terms, and therefore the sum is obtained by multiplying the average by the number of terms.

Example 7. Find the first term and the common difference of an A.P., if the fourth term is 23 and the tenth term is 110.

With the previous notation, we have

$$a+3d=23;\quad a+9d=110.$$

By subtraction, $6d=87$; $\quad\therefore\ d=\frac{87}{6}=14\frac{1}{2}$;

$$\therefore\ a=23-3d=23-43\tfrac{1}{2}=-20\tfrac{1}{2}.$$

$\therefore$ the first term is $-20\frac{1}{2}$ and the common difference is $14\frac{1}{2}$

Example 8. Find the sum of 40 terms of the A.P.,

$$5,\ 5\tfrac{2}{3},\ 6\tfrac{1}{3},\ 7,\ 7\tfrac{2}{3},\ \ldots .$$

With the previous notation, we have

$$a=5, \quad d=\tfrac{2}{3}, \quad n=40.$$

$$\therefore\ \mathrm{S}=\frac{n}{2}[2a+(n-1)d]$$
$$=\tfrac{40}{2}[10+39\times\tfrac{2}{3}]=20(10+26)$$
$$=20\times 36=720.$$

Harmonical Progression. If a series of numbers is such that their reciprocals are in A.P., the series is said to be in *harmonical progression*, for short, H.P.

Thus $\frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \frac{1}{14}, \ldots$ are in H.P., because $5, 8, 11, 14, \ldots$ are in A.P.; and $\frac{4}{9}, \frac{1}{3}, \frac{4}{15}, \frac{2}{9}, \frac{4}{21}, \ldots$ are in H.P., because $\frac{9}{4}, 3, \frac{15}{4}, \frac{9}{2}, \frac{21}{4}, \ldots$ are in A.P.

The *harmonic mean* of 2 numbers a, b is a number x such that a, x, b are in H.P.; $\therefore\ \frac{1}{a}, \frac{1}{x}, \frac{1}{b}$ are in A.P.;

$$\therefore\ \frac{1}{x}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{a+b}{2ab}; \quad \therefore\ x=\frac{2ab}{a+b}.$$

There is no formula for the sum of n terms of a harmonical progression.

EXERCISE V. d

1. Find the 10th term and 20th term of 7, 11, 15, 19, ...

2. Find the 12th term and 50th term of 120, 113, 106, 99, ...

Sum the following series in A.P., Nos. 3-11:

3. 10, 13, 16, 19, ... to 12 terms.

4. $7, 7\tfrac{3}{4}, 8\tfrac{1}{2}, 9\tfrac{1}{4}, \ldots$ to 33 terms.

5. 16, 12, 8, 4, 0, ... to 9 terms and to 20 terms.

6. $3, -5, \ldots$ to 10 terms.

7. 3·4, 2·8, ... to 16 terms.

8. $\frac{1}{3}, \frac{1}{2}, \ldots$ to 25 terms.

9. First term 10, last term 30; 8 terms.

10. First term 7, last term -13; 10 terms.

11. First term 3·5, last term 8; 16 terms.

12. How many terms are taken of the series 1, 2, 3, 4, 5, ..., if their sum is 120?

13. How many terms of the series 3, $4\frac{1}{2}$, 6, $7\frac{1}{2}$, ... must be taken to give a sum of 156 ?

14. Find the first term and the common difference of an A.P. if the 3rd term is 11 and the 7th term is 35.

15. The 5th term of an A.P. is 7 and the 11th term is -35; find the 1st term and the 7th term.

16. Find the sum of all the odd numbers between 100 and 200.

17. Find the simplest form for the nth term in the series :

$$\frac{1}{1}, \frac{1+3}{1+4}, \frac{1+3+5}{1+4+7}, \frac{1+3+5+7}{1+4+7+10}, \dots$$

18. A shop sells various sizes of tin kettles, and the prices of successive sizes rise by equal amounts. The smallest costs 2s. and the largest 8s.; it costs £2 10s. to buy one of every kind. How many kinds are there ? What is the cost of the smallest but one ?

19. Is 690 a term of the A.P., 4, 11, 18, 25, ... ?

20. A clerk's commencing salary is £100 a year; he is offered a choice between a yearly rise of £5 and a rise of £22 every 4 years. Calculate the total sum he will receive in the course of 33 years under each arrangement.

21. The nth term of a series is $3n+1$; find the sum of n terms.

22. From a piece of wire 5 ft. long, 25 pieces are cut off, each 0·1 in. longer than the preceding piece. If the wire is exactly used up, find the length of the first piece cut off.

23. I lay aside each year £20 more than I laid aside the year before, starting with £100 in the first year. How many years will it take me to lay aside £5800 ?

24. Find n if the sum of n terms of the series 2, 5, 8, 11, ... is equal to the sum of n terms of the series 47, 45, 43, 41, ...

25. Find the sum of all numbers less than 100 which are not divisible by 5.

26. Find the harmonic mean of (i) 9 and 12; (ii) $\frac{1}{p}$ and $\frac{1}{q}$.

27. The first two terms of a series in H.P. are 6, 3; find the 3rd term, the 4th term and the nth term.

[*For further practice, see Appendix, Ex. F.P. 5, p.* 157.]

Geometrical Progression

Start with any number, say, 7, multiply it by any other number, say, 3; then multiply again by 3, and so on; the successive numbers,

7, 21, 63, 189, 567. ... .

obtained in this way are said to be in **Geometrical Progression**, written, for short, **G.P.** The terms in a G.P. series can be written more simply by using the index notation; thus the series given above may be written,

$$7,\ 7.3,\ 7.3^2,\ 7.3^3,\ 7.3^4,\ \ldots \qquad \text{.....................(i)}$$

Again, start with 24 and multiply successively by $\frac{3}{2}$; this gives the G.P.,

$$24,\ 36,\ 54,\ 81,\ \tfrac{243}{2},\ \ldots .$$

But it is more easily recognised as a G.P. if we write it in the index form:

$$24,\ 24(\tfrac{3}{2}),\ 24(\tfrac{3}{2})^2,\ 24(\tfrac{3}{2})^3,\ 24(\tfrac{3}{2})^4,\ \ldots \qquad \text{.................(ii)}$$

Again, start with 18 and multiply successively by $\frac{2}{3}$; we then obtain the G.P.,

$$18,\ 12,\ 8,\ \tfrac{16}{3},\ \tfrac{32}{9},\ \tfrac{64}{27},\ \ldots ,$$

and this may be written,

$$18,\ 18(\tfrac{2}{3}),\ 18(\tfrac{2}{3})^2,\ 18(\tfrac{2}{3})^3,\ \ldots \qquad \text{...................(iii)}$$

The fundamental property of a G.P. is that the **ratio** of each term to the term preceding it is **constant.**

Thus the " constant ratio " of series (i) is 3;
the " constant ratio " of series (ii) is $\frac{3}{2}$;
the " constant ratio " of series (iii) is $\frac{2}{3}$.

If the constant ratio of a G.P. is negative, the terms will be alternately positive and negative.

Thus the numbers

$$12,\ -6,\ 3,\ -\tfrac{3}{2},\ \tfrac{3}{4},\ -\tfrac{3}{8},\ \ldots \qquad \text{.....................(iv)}$$

form a G.P. in which the constant ratio is $(-\frac{1}{2})$.

Example 9. Find the nth term in each of the series (i)-(iv) above.

(i) The nth term of the series $7,\ 7.3,\ 7.3^2,\ 7.3^3, \ldots$ is obviously 7.3^{n-1}.

(ii) The nth term of the series $24,\ 24(\frac{3}{2}),\ 24(\frac{3}{2})^2, \ldots$ is $24 . (\frac{3}{2})^{n-1}$; this may be written:

$$2^3 . 3 . \frac{3^{n-1}}{2^{n-1}} = \frac{3^n}{2^{n-4}}.$$

(iii) The nth term of the series $18,\ 18(\frac{2}{3}),\ 18(\frac{2}{3})^2, \ldots$ is $18 . (\frac{2}{3})^{n-1}$;

this equals $2 . 3^2 . \dfrac{2^{n-1}}{3^{n-1}} = \dfrac{2^n}{3^{n-3}}$.

(iv) The nth term of the series 12, $12(-\frac{1}{2})$, $12(-\frac{1}{2})^2$, ... is $12(-\frac{1}{2})^{n-1}$; this equals $2^2 . 3 . \frac{(-1)^{n-1}}{2^{n-1}} = (-1)^{n-1} . \frac{3}{2^{n-3}}$.

Example 10. Find the sum of 7 terms of the G.P.,

$$2, 6, 18, 54, \ldots .$$

Let S denote the required sum. The common ratio is 3.

$$\therefore\ S = 2 + 6 + 18 + 54 + 162 + 486 + 1458.$$

If we multiply each side by the common ratio 3, each term is changed into the term which used to follow it.

$$\therefore\ 3S = 6 + 18 + 54 + 162 + 486 + 1458 + 4374.$$

Therefore, if we subtract, all terms except the first and last disappear.

$$\therefore\ S - 3S = 2 - 4374; \quad \therefore\ 2S = 4372;$$

$$\therefore\ S = 2186.$$

It saves time to use the index notation throughout. Thus

$$S = 2 + 2 . 3 + 2 . 3^2 + 2 . 3^3 + 2 . 3^4 + 2 . 3^5 + 2 . 3^6;$$

$$\therefore\ 3S = \quad 2 . 3 + 2 . 3^2 + 2 . 3^3 + 2 . 3^4 + 2 . 3^5 + 2 . 3^6 + 2 . 3^7.$$

Subtract:

$$\therefore\ -2S = 2 - 2 . 3^7; \quad \therefore\ 2S = 2 . 3^7 - 2;$$

$$\therefore\ S = 3^7 - 1.$$

Geometric Mean. If we take any 3 numbers in G.P., the middle term is called the **Geometric Mean** between the two outside terms.

For example, 9, 15, 25 are numbers in G.P. because $\frac{15}{9}$ equals $\frac{25}{15}$, each being $\frac{5}{3}$.

$\therefore$ 15 is the geometric mean between 9 and 25.

Since the numbers 9, -15, 25 are also in G.P. (common ratio $-\frac{5}{3}$), it follows that -15 is also the geometric mean between 9 and 25, but it is customary to consider only positive values.

If x, y, z are 3 numbers in G.P., the common ratio is given by $\frac{y}{x}$ and by $\frac{z}{y}$;
$$\therefore\ \frac{y}{x} = \frac{z}{y}; \quad \therefore\ y^2 = xz.$$

$$\therefore\ y = \pm\sqrt{(xz)},$$

and we say that *the geometric mean between x and z is* $+\sqrt{(xz)}$.

Hence the geometric mean between two numbers is the (positive) square root of their product.

EXERCISE V. e

1. Find a simple form for the nth term of each of the following series. If the series is a G.P., write down the constant ratio.

(i) $5, 25, 125, 625, \ldots$; (ii) $3^2, 3^5, 3^8, 3^{11}, \ldots$;

(iii) $10^3, 10^6, 10^{12}, 10^{24}, \ldots$;

(iv) $40, 60, 90, 135, \ldots$; (v) $3^2, 6^2, 12^2, 24^2, \ldots$;

(vi) $\frac{1}{4}, \frac{1}{6}, \frac{1}{9}, \ldots$; (vii) $27, -18, 12, -8, \ldots$;

(viii) $2^2, 4^2, 6^2, 8^2, \ldots$; (ix) $\frac{2}{3}, \frac{4}{6}, \frac{6}{12}, \frac{8}{24}, \ldots$;

(x) $a^2, -a, 1, -\dfrac{1}{a}, \ldots$.

2. Write down the 3rd term, 4th term and nth term of a G.P., whose first two terms are as follows:

(i) $5, 10$; (ii) $10, 5$; (iii) $6, 8$; (iv) $6, -8$;

(v) $1, -1$; (vi) $-1, 1$; (vii) $3, \frac{1}{3}$; (viii) $72, -54$.

State *in index form* the last term in the following series in G.P.:

3. $24, 12, 6, \ldots$ to 7 terms. **4.** $\frac{1}{2}, -\frac{1}{6}, \frac{1}{18}, \ldots$ to 8 terms.

5. $a, ar, ar^2, \ldots$ to 10 terms. **6.** $a^2, ab, b^2, \ldots$ to 15 terms.

7. $3, -3, 3, \ldots$ to n terms. **8.** $36, -24, 16, \ldots$ to n terms.

Find the sum of the following series in G.P.; *leave the answer in index form.*

9. $5, 20, 80, \ldots$ to 7 terms. **10.** $3, 6, 12, \ldots$ to 8 terms.

11. $4, 12, 36, \ldots$ to 9 terms. **12.** $4, -12, 36, \ldots$ to 9 terms.

13. $10, 5, 2\frac{1}{2}, \ldots$ to 12 terms. **14.** $1, \frac{1}{2}, \frac{1}{4}, \ldots$ to 10 terms.

15. $24, 36, 54, \ldots$ to 8 terms. **16.** $144, -108, 81, \ldots$ to 7 terms.

17. Find the Geometric Mean between (i) 12 and 75; (ii) 2 and $\frac{1}{2}$.

18. The 1st and 4th terms of a G.P. are 24 and 81. Find the 2nd and 3rd terms.

General Method

Suppose the first term of a G.P. is a and that the common ratio is r, then the terms of the series are

$$a, ar, ar^2, ar^3, \ldots.$$

In the 3rd term, the power to which r occurs is 2;

in the 4th term, the power to which r occurs is 3;

and so on.

$\therefore$ the nth term is $\mathbf{ar^{n-1}}$.

Sum of a G.P. Let the sum of n terms be S.

Then $\quad S = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}.$

Multiply each side by the common ratio r; each term is then changed into the term which used to follow it.

$$\therefore \; rS = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + ar^n.$$

$\therefore$ if we subtract, all terms except the first and last disappear.

$$\therefore \; S - rS = a - ar^n; \quad \therefore \; S(1-r) = a(1-r^n);$$

$$\therefore \; S = \frac{a(1-r^n)}{1-r}.$$

This is the most convenient form for S if r is less than 1; but if r is greater than 1, the numerator and denominator would each be negative; in this case, S is written in the form,

$$S = \frac{a(r^n - 1)}{r - 1}.$$

The general formula is therefore

$$S = \frac{a(r^n - 1)}{r-1} = \frac{a(1-r^n)}{1-r}.$$

Example 11. Find the sum of 12 terms of the G.P.,

$$6\tfrac{3}{4}, \; -4\tfrac{1}{2}, \; 3, \; -2, \; \ldots .$$

Writing the series in the form, $\frac{27}{4}, \; -\frac{9}{2}, \; 3, \; -2, \ldots$, we see that the terms are in G.P., with common ratio $-\frac{2}{3}$.

$\therefore$ with the previous notation,

$$a = \tfrac{27}{4}, \; r = -\tfrac{2}{3}, \; n = 12.$$

$$\therefore \; S = \frac{a(1-r^n)}{1-r} = \frac{\frac{27}{4}[1 - (-\frac{2}{3})^{12}]}{1-(-\frac{2}{3})}$$

$$= \frac{\frac{27}{4}[1-(\frac{2}{3})^{12}]}{1\frac{2}{3}}, \text{ since } (-1)^{12} = +1$$

$$= \tfrac{27}{4} \times \tfrac{3}{5}[1 - (\tfrac{2}{3})^{12}] = \frac{81}{20} - \frac{3^4}{2^2 . 5} \cdot \frac{2^{12}}{3^{12}}$$

$$= 4\tfrac{1}{20} - \frac{2^{10}}{5 . 3^8}.$$

An approximate value of this expression may be found by using logarithms. With 4-figure tables, we find $\dfrac{2^{10}}{5 . 3^8} \simeq 0{\cdot}031$ (only 2 significant figures are reliable).

$\therefore$ the sum of 12 terms $\simeq 4{\cdot}05 - 0{\cdot}031 = 4{\cdot}019$.

Example 12. Insert 2 geometric means between 320 and 135.

Another way of stating this is as follows :

Find numbers x and y such that

$$320, \quad x, \quad y, \quad 135$$

are in G.P.

Let the common ratio be r.

Then $x = 320r$, $y = 320r^2$, $135 = 320r^3$;

$$\therefore\ r^3 = \tfrac{135}{320} = \tfrac{27}{64}; \quad \therefore\ r = \sqrt[3]{\tfrac{27}{64}} = \tfrac{3}{4}.$$

$$\therefore\ x = 320 \times \tfrac{3}{4} = 240\ ; \ y = 240 \times \tfrac{3}{4} = 180.$$

$\therefore$ the required geometric means are 240, 180.

Check : $\frac{240}{320} = \frac{3}{4}$, $\frac{180}{240} = \frac{3}{4}$, $\frac{135}{180} = \frac{3}{4}$.

EXERCISE V. f

Sum the following series in G.P., leaving the answer in index form.

1. $6, 24, 96, \ldots$ to 10 terms.
2. $8, -24, 72, \ldots$ to 12 terms.
3. $4, 2, 1, \ldots$ to 15 terms.
4. $2{\cdot}5, 1, 0{\cdot}4, \ldots$ to 20 terms.
5. $1, -a^2, a^4, \ldots$ to 15 terms.
6. $2b, 6b^2, 18b^3, \ldots$ to n terms.
7. $c, 1, \dfrac{1}{c}, \ldots$ to k terms.
8. $2^2, 4^2, 8^2, \ldots$ to $(n-1)$ terms.
9. Insert 2 geometric means between 54 and 2.
10. Insert 3 geometric means between $3\frac{5}{9}$ and 18.
11. Using 4-figure logarithms, find an approximate value of

    (i) $10\{1 + (1.04) + (1.04)^2 + \ldots + (1.04)^{10}\}$;

    (ii) $12\left\{1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \ldots + \dfrac{1}{2^8}\right\}$.

12. The nth term of a series is $2^{n-1} + 2n$. Write down the first 4 terms. Find the sum of the first r terms.

13. How many terms of the G.P., 4, 6, 9, ... , must be taken to give a sum greater than 8000 ?

14. The 3rd term of a G.P. is 40 and the 6th term is 625 ; find the 1st term.

15. Use 4-figure logarithms to find an approximate value of

$$3 + 2^{\frac{1}{4}} . 3^{\frac{3}{4}} + 2^{\frac{1}{2}} . 3^{\frac{1}{2}} + 2^{\frac{3}{4}} . 3^{\frac{1}{4}} + \ldots \text{ to 12 terms.}$$

16. Use 4-figure logarithms to find an approximate value of the sum (i) to 9 terms, (ii) to 10 terms, of the G.P.,

$$500, \ -400, \ 320, \ -256, \ \ldots .$$

Compound Interest and Annuities

The most important practical application of series in G.P. is in connection with annuities and insurance.

It was shown on p. 56 that the amount of £P invested at r per cent. per annum compound interest for n years is £A, where

$$A = P\left(1 + \frac{r}{100}\right)^n = P \,.\, R^n, \text{ if } R = 1 + \frac{r}{100}.$$

Present Value. If a payment of £A is due in n years' time, we can easily find the present value, if compound interest is reckoned as before.

If the present value is £P, the amount after n years is £P . R^n, where as before $R = 1 + \frac{r}{100}$;

$$\therefore\ P.R^n = A; \quad \therefore\ P = \frac{A}{R^n} = A \,.\, R^{-n}.$$

$$\therefore \text{ the present value is } £(A \,.\, R^{-n}).$$

The difference, $£A - £P = £A(1 - R^{-n})$, is called the *Discount.*

Annuities. To find the purchase price of an annuity of £A, if there are in all n annual payments, the first being made in one year's time, allowing r per cent. per annum compound interest.

As before, put $1 + \frac{r}{100} = R$.

The first payment, £A in 1 year's time, is worth $£\frac{A}{R}$ now.

The second payment, £A in 2 years' time, is worth $£\frac{A}{R^2}$ now; and so on.

$\therefore$ the whole annuity is worth now

$$£\left(\frac{A}{R} + \frac{A}{R^2} + \frac{A}{R^3} + \dots + \frac{A}{R^n}\right)$$

$$= £\frac{\frac{A}{R}\left(1 - \frac{1}{R^n}\right)}{1 - \frac{1}{R}} = £\frac{A(1 - R^{-n})}{R - 1}.$$

Note. If the annuity is *deferred*, *i.e.* if the first payment takes place after k years, and if n payments are to be made in all, its purchase price now would be

$$£\left(\frac{A}{R^k} + \frac{A}{R^{k+1}} + \dots + \frac{A}{R^{k+n-1}}\right) = £\frac{A(1 - R^{-n})}{R^{k-1}(R - 1)}.$$

No attempt should be made to commit these results to memory. If the method is understood, there is no difficulty in applying it to numerical examples.

In general, 4-figure tables do not give a sufficient degree of accuracy: the following extracts from 7-figure tables should be used, where appropriate:

Number.	Logarithm.	Number.	Logarithm.
1·025	0·0107239	1·04	0·0170333
1·03	0·0128372	1·045	0·0191163
1·035	0·0149403	1·05	0·0211893

EXERCISE V. g

1. (i) What is the amount of £100 in 10 years' time, allowing 4 per cent. compound interest?

   (ii) A man deposits £100 annually to accumulate at 4 per cent. per annum compound interest. How much will he have standing to his credit just after he has made the tenth deposit?

2. A man pays £100 at the beginning of each year into a pension fund, and it is agreed that at the end of 15 years he will receive back what he has paid together with $4\frac{1}{2}$ per cent. per annum compound interest. What should he receive?

3. On January 1st of each year from 1915 to 1925 inclusive, a man invested £100 at 5 per cent. per annum compound interest. What was the value of his investments on December 31, 1925?

4. What lump sum was paid on January 1st, 1920, in order to secure a payment of £100 a year for ten years, the first payment being made on January 1st, 1921, allowing 5 per cent. per annum compound interest?

5. A man buys a life annuity of £200 a year to be paid yearly, starting in 1 year's time. What lump sum will the Insurance Company require if they expect that 12 payments will be necessary, allowing $2\frac{1}{2}$ per cent. per annum compound interest?

6. A Corporation borrows £10,000 and repays it by 30 equal annual payments, the first being made one year after the loan has been raised. Allowing compound interest at 5 per cent. per annum, calculate the amount of each annual payment.

7. Repeat No. 6 if the loan is repaid by 60 equal half-yearly payments, the first being made 6 months after the loan has been raised, allowing $2\frac{1}{2}$ per cent. half-yearly compound interest.

8. A man leaves property worth £2000 a year, paid half yearly, to his wife for her life-time, and after her death to a hospital. His wife's expectation of life is 8 years; what is the present value of

the legacy which she receives? Allow 3 per cent. half yearly compound interest and assume the first payment is due in 6 months time.

9. Find to the nearest £10 the present value of a deferred annuity of £100 a year for 20 years, if the first payment is due in 5 years' time; compound interest, 4 per cent. per annum.

10. On January 1st, 1930, a man buys a deferred annuity for £5000. The first payment is to be made on January 1st, 1935, and additional equal annual payments are due up to January 1st, 1950, inclusive. What is the amount of each annual payment, reckoning compound interest at $3\frac{1}{2}$ per cent. per annum?

Series with Limiting Sums

Draw a straight line AB 2 inches long. Bisect it at P_1; then bisect P_1B at P_2, and bisect P_2B at P_3, and so on.

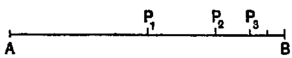


FIG. 5.

$AP_1 = 1$ in.; $AP_2 = (1 + \frac{1}{2})$ in.; $AP_3 = (1 + \frac{1}{2} + \frac{1}{4})$ in.; and so on.

After the process has been repeated n times, we have

$$AP_n = \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}}\right) \text{in.}$$

The method of construction shows that, however often the process is repeated, the point P_n remains on the left of B and the length of AP_n remains less than the length AB, 2 in.

In other words, however many terms of the series,

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots,$$

we add up, the sum always remains less than 2.

But by repeating the process sufficiently often we can make the point P_n approach as near B as we like, since $P_nB = \frac{1}{2^{n-1}}$ in., and this length can be made as small as we please by taking n sufficiently large.

$\therefore$ by taking a sufficient number of terms of the series,

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots,$$

we can obtain a sum which for this number of terms, and for every greater number of terms, is as near 2 as we please.

We therefore call 2 the **limiting sum** of the series,

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots .$$

Now apply the formula to the same series. Let s_n denote the sum of n terms of the series.

Then $s_1 = 1$, $s_2 = 1 + \frac{1}{2}$; $s_3 = 1 + \frac{1}{2} + \frac{1}{2^2}$; and so on.

$$\begin{aligned}\therefore\ s_n &= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}} \\ &= \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = \frac{1 - (\frac{1}{2})^n}{\frac{1}{2}} \\ &= 2\left(1 - \frac{1}{2^n}\right) = 2 - \frac{1}{2^{n-1}}.\end{aligned}$$

However large n is, s_n is always less than 2. But by making n large enough, $\frac{1}{2^{n-1}}$ can be made as small as we please.

We therefore say that, when n tends to infinity, $\frac{1}{2^{n-1}}$ tends to 0, and s_n tends to 2.

This is written in the following way:

When $\qquad n \to \infty, \ \frac{1}{2^{n-1}} \to 0$ and $s_n \to 2$.

The phrase "n tends to infinity" merely means that n is allowed to take values greater than any stated number, however large.

Next consider the series,

$$3, 6, 12, 24, \dots .$$

The sum of n terms $= \dfrac{3(2^n - 1)}{2 - 1} = 3 \cdot 2^n - 3$.

By taking n large enough, $3 \cdot 2^n$ can be made as large as we please; therefore this series has no limiting sum.

If the common ratio of a G.P. is 1, the sum s_n of n terms is

$$s_n = a + a + a + \dots + a = na.$$

And, if the common ratio is greater than 1, then $s_n > na$. But by taking n large enough, na can be made as large as we please; therefore a G.P. for which $r = 1$ or $r > 1$ has no limiting sum.

From the formula, if $r > 1$, we have $s_n = \dfrac{a(r^n - 1)}{r - 1}$. But by taking n large enough, we have just proved that s_n can be made as large as we please, and therefore it follows that, if $r > 1$, r^n

can be made as large as we please; and so $\frac{1}{r^n}$ or $\left(\frac{1}{r}\right)^n$ can be made as small as we please, if $\frac{1}{r}$ lies between 0 and 1.

$\therefore$ if k is *any* fraction between 0 and 1, k^n can be made as small as we please by taking n large enough.

But $s_n = a + ak + ak^2 + \dots + ak^{n-1} = \frac{a(1-k^n)}{1-k}$;

$$\therefore\ s_n = \frac{a}{1-k} - \frac{a}{1-k} \cdot k^n.$$

$\therefore$ if k is any fraction between 0 and 1 (or between 0 and -1), by taking n large enough, $\frac{a}{1-k} \cdot k^n$ can be made as small as we please, and so s_n becomes, and remains, as near $\frac{a}{1-k}$ as we please.

$\therefore$ if k is any fraction between -1 and $+1$, the series

$$a,\ ak,\ ak^2,\ ak^3, \dots$$

has a limiting sum equal to $\frac{a}{1-k}$.

This result may be written as follows:

If $-1 < r < 1$ and if $s_n = a + ar + ar^2 + \dots + ar^{n-1}$, then, when $n \to \infty$,

$$s_n \to \frac{a}{1-r}.$$

If $r = -1$, the series is $a,\ -a,\ a,\ -a, \dots$, and therefore $s_n = a$ if n is odd and $s_n = 0$ if n is even, so there is no unique limiting sum.

If $r < -1$, *e.g.* in the series $2,\ -6,\ 18,\ -54, \dots$, the value of s_n when n is odd is of opposite sign to its value when n is even; and the *numerical* value of s_n increases indefinitely as n increases; the series has therefore no limiting sum.

The "limiting sum" of a series is generally called its **sum to infinity.**

Example 13. What is the meaning of $0{\cdot}\dot{6}$. Express it as a vulgar fraction.

$$0{\cdot}6 = \frac{6}{10};\ 0{\cdot}66 = \frac{6}{10} + \frac{6}{10^2};\ 0{\cdot}666 = \frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3};$$

and so on.

$0{\cdot}\dot{6}$ means the limiting sum of the series,

$$\frac{6}{10},\ \frac{6}{10^2},\ \frac{6}{10^3}, \dots.$$

We know that there is a limiting sum, because the common ratio $\frac{1}{10}$ is less than 1.

$$\therefore \text{ the limiting sum} = \frac{\frac{6}{10}}{1-\frac{1}{10}} = \frac{6}{10} \div \frac{9}{10} = \frac{2}{3}.$$

$$\therefore\ 0{\cdot}\dot{6} = \frac{2}{3}.$$

EXERCISE V. h

1. By using logarithms, find the least integral value of n for which (i) $3^n > 1{,}000{,}000$; (ii) $(\frac{3}{4})^n < 0{\cdot}0001$.

2. Express in decimals correct to 3 significant figures the sum of 1, 2, 3, 4, 5 terms and the limiting sum of the series.

(i) $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \ldots$; (ii) $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}$;
(iii) $\frac{1}{2}, \frac{1}{10}, \frac{1}{50}, \ldots$; (iv) $\frac{1}{2}, -\frac{1}{6}, \frac{1}{18}, -\frac{1}{54}, \ldots$.

3. By using logarithms, find approximately the difference between $\dfrac{a}{1-r}$ and $\dfrac{a(1-r^n)}{1-r}$, if

(i) $a = \frac{1}{2}$, $r = 1{\cdot}05$, $n = 1000$; (see p. 76)
(ii) $a = 10$, $r = 0{\cdot}2$, $n = 100$.

4. Find the least number of terms whose sum differs by less than $0{\cdot}001$ from the limiting sum, for the G.P., 12, 9, $6\frac{3}{4}$, ...

Find the limiting sums of the following series in G.P.:

5. 12, 4, $1\frac{1}{3}$, ... 6. 12, 8, $5\frac{1}{3}$, ...

7. 24, -12, 6, -3, ... 8. $\frac{9}{10}, \frac{9}{100}, \frac{9}{1000}, \ldots$.

9. 25, -15, 9, $-5{\cdot}4$, ... 10. 1, $(1{\cdot}05)^{-1}$, $(1{\cdot}05)^{-2}$, ...

11. Express as a vulgar fraction, (i) $0{\cdot}\dot{2}\dot{7}$; (ii) $0{\cdot}4\dot{5}$.

12. How many terms of the series, 1, $\dfrac{1}{1{\cdot}04}$, $\dfrac{1}{(1{\cdot}04)^2}$, ..., must be taken to give a sum which differs from the limiting sum by less than $0{\cdot}1$?

13. A pendulum is set swinging ; its first oscillation is through 18°, and each succeeding oscillation is $\frac{5}{6}$ of the one before it. What is the total angle through which it oscillates before it stops ?

14. The height of a tree was 45 feet ; and it increased by 5 feet in the course of the next year. If in each succeeding year, the growth is $\frac{7}{8}$ of that in the previous year, find the limiting height.

15. An elastic ball, dropped from a height of 16 ft., takes 1 sec. to reach the ground ; the first bounce up and down takes $1\frac{1}{2}$ sec., and each following one takes $\frac{3}{4}$ as long as the bounce before it. How long is it from the time the ball is dropped to when it stops bouncing ?

16. (i) Use the formula on p. 75 to find what single payment on January 1st, 1930, will secure a payment of £100 a year on every succeeding January 1st for ever, allowing 4 per cent. per annum compound interest ?

(ii) What is the annual simple interest on this lump sum at 4 per cent. per annum ?

[*For further general practice on series, see Appendix, Ex. F.P. 5, p.* 157.]

EXERCISE V. j

Miscellaneous Progressions

Sum the series in Nos. 1-12 :

1. $\frac{4}{3}, \frac{23}{12}, \frac{5}{2}, \ldots$ 25 terms.
2. $\frac{3}{2}, -\frac{9}{8}, \frac{27}{32}, \ldots$ 8 terms.
3. $2, -\frac{3}{2}, \frac{9}{8}, \ldots$ 9 terms.
4. $\frac{5}{3}, \frac{9}{4}, \frac{17}{6}, \ldots$ 23 terms.
5. $\frac{8}{3}, -\frac{8}{9}, \frac{8}{27}, \ldots$ 10 terms, to 2 places of decimals.
6. $2{\cdot}4, 3{\cdot}2, 4, \ldots$ 30 terms.
7. $8, -4, 2, -1, \ldots$ to ∞.
8. $\frac{1}{3}, -\frac{1}{2}, \frac{3}{4}, \ldots$ 8 terms.
9. $8, 20, 50, \ldots$ 7 terms.
10. $2\frac{1}{2}, -1\frac{1}{4}, \frac{5}{8}, \ldots$ 20 terms, to 2 places of decimals.
11. $-7, -5\frac{2}{3}, -4\frac{1}{3}, \ldots$ 21 terms.
12. $1, -0{\cdot}4, 0{\cdot}16, \ldots$ to ∞.
13. Evaluate $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+\ldots$ to n terms.
14. Evaluate $(a+x)^2+(a^2+x^2)+(a-x)^2+\ldots$ to n terms.
15. Insert 3 Arithmetic Means between 117 and 477.
16. Insert 2 Harmonic Means between 2 and 4.

17. The sum of 11 terms of an A.P. is 22 and the common difference is $\frac{3}{5}$. Find the first term.

18. Find the sum of the first n terms of an A.P. whose pth term is a and qth term is b.

19. A contractor undertakes to bore a well at 30s. for the first 10 feet, 35s. for the next 10 feet, 40s. for the next 10 feet, and so on. What is the depth of a well that costs £105 ?

20. How many terms must be taken of a G.P. whose first term is $1{\cdot}5$ and common ratio $0{\cdot}5$ in order that their sum may differ from 3 by less than $0{\cdot}0001$?

21. The first term of an A.P. is 24 and the third term is 4, how many terms are required to give a sum of 20 ?

22. A clerk is engaged at a salary of £120 a year, to be increased £12 a year at the end of each year. He saves one-quarter of his salary every year ; find the total saved at the end of 10 years.

23. The third and fourth terms of an A.P. are 15 and $13\frac{1}{2}$; how many terms will make the sum zero ?

24. Find the sum of all integral powers of 2 between 100 and 1,000,000. Give the answer in index form.

25. £500 is invested at 5 per cent. per annum compound interest. After how many complete years will the amount first exceed £700 ?

26. Find to the nearest £ the present value of an annuity of £60 a year for 25 years at 5 per cent. per annum compound interest, the first payment being made in 1 year's time.

[*For additional examples on series, see Appendix, Ex. T.* 5, *p.* 172.]

TEST PAPERS C. 11-20

C. 11

1. Simplify (i) $8^{-\frac{2}{3}}+4^{-1}-6^{-1}$; (ii) 10^0-10^{-1};

$$\text{(iii) } 3x^{\frac{1}{3}}y^{\frac{2}{3}}\times(3x^{-\frac{1}{3}}y)^{-2}.$$

2. Given $\log 2 \simeq 0{\cdot}3010$, find, without using tables, approximate values of $\log 8$, $\log 5$, $\log 3{\cdot}2$. (Base 10.)

3. Solve (i) $4{\cdot}7^x=17{\cdot}2$; (ii) $y-6\sqrt{y}=16$.

4. What is (i) the number of terms, (ii) the sum of the following series : 23, 29, 35, 41, 47, , 125 ?
Prove that if n is any positive integer, 10^n+1 belongs to this A.P.

5. A man and a boy pace a distance of 60 yards together. The boy's paces are 6 inches shorter than the man's, and he takes 12 more paces for the distance. What is the length of the boy's pace ?

C. 12

1. Simplify (i) $9^{2n}\div 3^n$; (ii) $4^p.5^q.10^{p-q}$.

2. Express in their simplest forms, using positive indices only :

$$\text{(i) } (16a^{-4}b^2)^{-\frac{1}{2}}; \qquad \text{(ii) } \frac{\sqrt[3]{(x^2y^{-4})}}{\sqrt{(x^{\frac{2}{3}}y^{\frac{4}{3}})}}.$$

3. Simplify (i) $\log 14+2\log 2-\log 8$; (ii) $\log 125\div\log 5$;
(iii) $\log_2 16$; (iv) $\log_2 8\times\log_3 2$

4. Sum the series :

(i) $2+4\frac{1}{2}+7+9\frac{1}{2}+\ldots$ 20 terms ;
(ii) $14+11+8+5+\ldots$ 10 terms.

5. How many terms of the G.P., 7, $3\frac{1}{2}$, $1\frac{3}{4}$, $\frac{7}{8}$ ..., must be taken, in order that the sum may differ from 14 by less than $0{\cdot}01$?

C. 13

1. Simplify (i) $a^{\frac{2}{3}} \times a^{\frac{7}{6}} \div a^{\frac{1}{2}}$; (ii) $(\frac{1}{4})^{-\frac{1}{2}} \times (\frac{8}{27})^{-\frac{2}{3}}$.

2. What is x if, for base 10,

(i) $\log x = 1 + \log 7$; (ii) $\log x = 2 - \log 4$;
(iii) $\log \sqrt{x} = 3$; (iv) $\log x = 3 \log 2 - 2 \log 3$?

3. Sum the series :

(i) $3 + 6 + 9 + 12 + \ldots$, 30 terms ;
(ii) $3 + 6 + 12 + 24 + \ldots$, 30 terms.

Use logarithms to evaluate the sum of (ii) correct to 2 figures.

4. If $v \propto l^3$ and $a \propto l^2$, find the variation relation between v and a.

What is the percentage increase in v when a increases by 21 per cent.

5. The 1st, 4th and 8th terms of an A.P. are in G.P., and the 1st term is 9. Find the 2nd term of the A.P. and the common ratio of the G.P.

C. 14

1. Simplify (i) $64^{\frac{1}{3}} \times (\frac{4}{3})^{-2} \times 5^0 \times 9^{-\frac{1}{2}}$; (ii) $4^m \times 6^{m-n} \div 12^{2m-n}$.

2. Write down the logarithms to base 9 of

(i) 81 ; (ii) 3 ; (iii) 27 ; (iv) $\frac{1}{9}$; (v) 27^n.

3. (i) Multiply $a^{\frac{1}{2}} + \sqrt{3} + 2a^{-\frac{1}{2}}$ by $a^{\frac{1}{2}} - \sqrt{3} + 2a^{-\frac{1}{2}}$.

(ii) Solve $x = 7x^{\frac{1}{2}} + 18$.

4. (i) Find the sum to 16 terms of $100 + 90 + 80 + 70 + \ldots$;

(ii) Find the limit sum of the G.P., $100 + 90 + 81 + \ldots$.

5. A boat's crew make their boat travel 10 yards for every stroke they take. In travelling 1600 yards, they can reduce the time taken by 20 seconds if they increase their rate of stroke by 2 per minute without decreasing the length travelled per stroke. How many strokes do they take per minute at the faster speed ?

C. 15

1. Simplify (i) $(x^3y^{-1})^{-2} \times (x^{-1}y^{-2})^{-1}$;

(ii) $(2\sqrt{6} + 5\sqrt{2})^2 - (2\sqrt{6} - 5\sqrt{2})^2$.

2. (i) Simplify $\log 2 \div \log 0{\cdot}5$;

(ii) Find to 3 figures the value of x, if $2^{x+2} = 3^x$.

3. (i) Make y the subject of the formula, $\log_{10}x + 2\log_{10}y = 6$;

(ii) Solve $x + y = 2$, $x^2 + 4y = 2y^2$.

4. Sum to $2n$ terms, n being an integer,

(i) $(a-1)+(3a-2)+(5a-3)+(7a-4)+\ldots$.

(ii) $(1+1)+b(b-c)+b^2(b^2+c^2)+b^3(b^3-c^3)+\ldots$.

5. A ball dropped from a height of 6 ft. bounces to a height of 4 ft. after its first impact, and in each successive bounce rises to two-thirds of the height of the previous bounce. How far does it travel altogether before it finally comes to rest ?

C. 16

1. Simplify (i) $\dfrac{12 \times 18^n}{6^{2n+1} \times (0 \cdot 5)^n}$; (ii) $\log\dfrac{p}{q}+\log\dfrac{q}{p}$.

2. (i) Expand $(x^{\frac{1}{2}}+x^{\frac{1}{3}})(x^{\frac{1}{6}}-1)$.

(ii) Divide $2a+3a^{\frac{1}{2}}b^{-1}+b^{-2}$ by $a^{\frac{1}{2}}+b^{-1}$.

3. (i) If $2\log a-\log b=\log c$, find a in terms of b, c.

(ii) Evaluate $\log_3 36-2\log_3 2$.

4. Prove that $\dfrac{401+403+405+\ldots+499}{1+3+5+\ldots+99}=9$, and show that $(101+103+105+\ldots+199)$ is a geometric mean between the numerator and the denominator.

5. If two unequal weights connected by a light string are hung over a smooth pulley, the tension in the string during the motion is equal to the harmonic mean between the weights. What is the tension when the weights are (i) 4 lb. and 6 lb., (ii) $(P+Q)$ lb. and $(P-Q)$ lb. ?

C. 17

1. Simplify (i) $(a^{-\frac{1}{2}}b^{\frac{3}{2}})^{-2}$; (ii) $\dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{3}-1}$.

(iii) $\log ab+\log bc-\log ca$; (iv) $\log\frac{1}{4}\div\log\frac{1}{8}$.

2. Evaluate to 3 figures $\log_2 5$ and $\log_5 2$. What is the connection between their values ?

3. Given that $x^{\frac{2}{3}}y^{\frac{1}{5}}z^{-\frac{4}{5}}=0 \cdot 35$, express x in the form cy^pz^q, giving c correct to 2 figures.

Find to 2 figures the value of x if $y=16$ and $z=32$.

4. How many terms of the G.P. 2, 3, $4\frac{1}{2}$, ... must be taken to obtain a sum exceeding 10,000 ?

5. The sum of n terms of an A.P. is $5n^2-11n$ for all values of n. (i) What is the common difference ? (ii) What is the sum of $2n$ terms ?

C. 18

1. Find to 3 figures the value of $\dfrac{(0{\cdot}4678)^{\frac{2}{3}}}{(3{\cdot}592)^2-(0{\cdot}592)^2}$.

2. Simplify (i) $\sqrt{\left(\dfrac{x^3}{y^{\frac{1}{2}}}\right)} \div (x^{\frac{1}{3}}y^{-\frac{1}{6}})^{\frac{3}{2}}$;

(ii) $(\log a^x - \log a^y) \div (\log c^p - \log a^q)$.

3. Write the following without using logarithmic notation :

(i) $\log_2 x + 3 \log_2 y = 4$;

(ii) $\log_9 x - \frac{1}{3} \log_9 y = \frac{1}{2}$.

4. There are 50 houses on one side of a street and the distance between each door and the next is 6 yards. A stupid coster leaves his barrow opposite house No. 1, and visits each house in turn, returning to his barrow between each visit. How far does he walk altogether ?

5. The Geometric Mean of two numbers p and q is 6 and their Harmonic Mean is 3·6. Find the values of p, q. Find also the nth term of the H.P. whose first 3 terms are p, 3·6, q. [Two answers.]

C. 19

1. Find the value of x if

(i) $2^x \times 4^{2x-1} = 8^x$; (ii) $(\sqrt{3})^x = \dfrac{1}{27^{x-1}}$; (iii) $(\sqrt{3})^x = 2^{x-1}$.

2. Simplify (i) $(3a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}})^2 \times (2abc)^{-\frac{1}{2}} \div \sqrt[3]{(8b)}$;

(ii) $\sqrt{(42-24\sqrt{3})}$.

3. (i) If $2{\cdot}37v^{1{\cdot}6} = 14{\cdot}8$, find the value of $\log_{10} v$.

(ii) Find x if $\log x \,.\, \log 27 = \log 8 \,.\, \log 9$.

4. The first term of a series is 6 and the second term is 8. Find its nth term supposing that the series is (i) an A.P., (ii) a G.P., (iii) an H.P.

Sum to n terms the series :

$$\log 8 + \log 12 + \log 18 + \log 27 + \dots .$$

5. The maximum velocity attainable by a car varies directly as its horse-power and inversely as the resistance. The resistance varies directly as the square of the velocity. Find how the velocity varies with the horse-power.

What is the percentage increase in the velocity if the horse-power is increased by 40 per cent. ?

C. 20

1. Find to 3 figures the value of

$$\text{(i) } \log_2 17\,; \quad \text{(ii) } 2^{-1\cdot 27}\,; \quad \text{(iii) } 10^{\log_{10} 3}.$$

2. Simplify (i) $\dfrac{a^3}{(a+b)^{\frac{1}{2}}} - \dfrac{a^2}{(a+b)^{-\frac{1}{2}}}$;

(ii) $\log(\log c^9) - \log(\log c^3)$.

3. Solve (i) $5^x \times 6^{x+2} = 8^{2x+1}$ (to 3 figures).

(ii) $\sqrt{(x+y)} + \sqrt{(x-y)} = \sqrt{(2x)}$, $x - y = 4$.

4. A man saves £300 every year. If he is able to invest his savings at the end of each year at 4 per cent. compound interest, how much will he have saved at the end of 20 years ?

5. An island had at a certain period a population of 200,000; and the population was increasing in G.P., the common ratio being 1·05 for 5-yearly intervals. The island was at this time producing food for 300,000 persons, and its produce was increasing in A.P. at 4000 units per year, the unit being 1 year's supply for one person, any surplus food being exported each year. Show that the island is still self-supporting after 155 years, but after 160 years will have ceased to be so.

[*For additional test papers, see Appendix*, *Z*. 11-20, *p*. 188.]

CHAPTER VI

RATIO AND PROPORTION

THE meaning and use of Ratio is one of the fundamental features of Arithmetic and enters into all geometrical problems which involve the idea of similarity.

If x and y are any two *numbers*, the ratio of x to y is measured by the fraction $\frac{x}{y}$ and is often written $x : y$; it represents the comparison between the two numbers. In the same way, two *quantities* can be compared with one another, *if they are of the same kind*. Thus the ratio of a feet to b yards is found by expressing each length in the *same unit*; the two lengths are $\frac{a}{3}$ yd. and b yd., or a ft. and $3b$ ft., or $12a$ in. and $36b$ in.; their ratio is represented by any one of the equal fractions, $\frac{\frac{a}{3}}{b}$ or $\frac{a}{3b}$ or $\frac{12a}{36b}$.

Two quantities of different kinds cannot be compared; there is no ratio between, say, 6 shillings and 8 hours.

The following notation is used for comparing three or more numbers with each other.

If x, y, z are three numbers, the relation

$$x : y : z = 7 : 3 : 15$$

means that $\frac{x}{y}=\frac{7}{3}$ and $\frac{y}{z}=\frac{3}{15}$, and so $\frac{x}{z}=\frac{7}{15}$. This is also written in the form, $\frac{x}{7}=\frac{y}{3}=\frac{z}{15}$.

Similarly, $x : y : z : v : w = a : b : c : d : e$ means that

$$\frac{x}{y}=\frac{a}{b},\ \frac{y}{z}=\frac{b}{c},\ \frac{z}{v}=\frac{c}{d},\ \frac{v}{w}=\frac{d}{e},$$

and so $\frac{v}{x}=\frac{d}{a}$, etc.; this is also written, $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{v}{d}=\frac{w}{e}$.

Example 1. If $a : b = 11 : 5$, find the ratio $(3a - 5b) : (a + 5b)$.

We may use either of the following methods:

(i) $$\frac{3a-5b}{a+5b} = \frac{\frac{3a}{b}-5}{\frac{a}{b}+5} = \frac{\frac{33}{5}-5}{\frac{11}{5}+5}$$

$$= \frac{8}{5} \div \frac{36}{5} = \frac{8}{36} = \frac{2}{9}.$$

(ii) Any two numbers in the ratio $11 : 5$ can be expressed in the form $11n$ and $5n$. Put $a = 11n$, then $b = 5n$.

$$\therefore \frac{3a-5b}{a+5b} = \frac{33n-25n}{11n+25n} = \frac{8n}{36n} = \frac{2}{9}.$$

Example 2. If $x : y = 4 : 5$ and $y : z = 3 : 7$, compare together x, y, z and find the ratio $(x + 2y) : 3z$.

$$\frac{x}{4} = \frac{y}{5} \text{ and } \frac{y}{3} = \frac{z}{7};$$

$$\therefore \frac{x}{12} = \frac{y}{15} \text{ and } \frac{y}{15} = \frac{z}{35};$$

$$\therefore \frac{x}{12} = \frac{y}{15} = \frac{z}{35}.$$

$$\therefore \mathbf{x : y : z} = 12 : 15 : 35.$$

Next, put $x = 12n$, then $y = 15n$ and $z = 35n$;

$$\therefore \frac{x+2y}{3z} = \frac{12n+30n}{3 \times 35n} = \frac{42n}{3 \times 35n} = \frac{2}{5}.$$

EXERCISE VI. a

1. What is the ratio of

(i) $18x$ shillings to £$(3x)$; (ii) $2b$ hours to $16c$ minutes;

(iii) y^2 sq. in. to z^2 sq. ft.; (iv) $8a^3$ cu. in. to b^3 cu. ft.?

2. If $a : b = 3 : 5$ and $c : d = 2 : 5$, find the following ratios:

(i) $a^2 : b^2$; (ii) $\frac{1}{a} : \frac{1}{b}$; (iii) $ac : bd$;

(iv) $ad : bc$; (v) $a^3 : b^3$; (vi) $\frac{a}{c} : \frac{b}{d}$.

3. Two numbers are in the ratio $a : b$. The first is n, what is the second? The second is k, what is the first?

4. Find the ratio of $x : y$,

(i) if $2x = 3y$; (ii) if $4x^2 = 9y^2$;

(iii) if $7x - 2y = x + 7y$; (iv) if $x = y + \frac{1}{4}y$;

(v) if x exceeds y by r per cent.

5. Compare together x, y, z

(i) if $x : y = 10 : 21$ and $y : z = 28 : 9$; (ii) if $x = 3y$ and $y = 5z$.

6. If $x : y = 5 : 3$, find the following ratios :

(i) $\dfrac{x-y}{x+y}$; (ii) $\dfrac{3x-y}{x+3y}$; (iii) $\dfrac{x^2+y^2}{x^2-y^2}$.

7. If $x : y : z = 6 : 10 : 15$, find the following ratios :

(i) $\dfrac{x}{y+z}$; (ii) $\dfrac{x+y}{y+2z}$; (iii) $\dfrac{3y-x}{x+2y+2z}$.

8. The ratio of the radii of two spheres is $2 : 5$. What is the ratio of (i) the areas of their surfaces, (ii) their volumes ?

9. The ratio of the heights of two circular cylinders is $3 : 4$, and the ratio of their diameters is $4 : 5$. What is the ratio of their volumes ?

10. The time of oscillation of a pendulum of length l feet is $1{\cdot}11\sqrt{l}$ seconds. The length of a pendulum is increased from $2{\cdot}4$ feet to 15 feet ; in what ratio is the time of oscillation altered?

11. With the data of No. 10, find the ratio in which the length of a pendulum must be altered to increase the time of oscillation by 10 per cent.

12. A river steamer takes 60 per cent. longer to make a journey up-stream than to go the same distance down-stream. Compare the speed of the steamer in still water with the rate of the current.

13. If $pv = 30$, find the ratio in which v alters when p is increased in the ratio $9 : 4$.

14. Find the ratio in which the following expressions are altered if x and y are each increased in the ratio $3 : 2$.

(i) $\dfrac{x}{y}$; (ii) $\dfrac{x^2}{y^2}$; (iii) $\dfrac{x+y}{x-y}$; (iv) $x+2y$;

(v) $\dfrac{x^2}{y}$; (vi) $\dfrac{x^3}{y^2}$; (vii) $\dfrac{x^2-y^2}{xy}$; (viii) $\dfrac{x^2y^2}{2x^3+3y^3}$.

15. Find the two possible values of the ratio of $x : y$ if

(i) $x^2 - 7xy + 12y^2 = 0$; (ii) $6x^2 + 11xy = 10y^2$.

16. What is the ratio of $a : b$ if $\left(\dfrac{a+3b}{a-2b}\right)^2 = 9$.

17. If $\dfrac{x+5y}{3x+y} = \dfrac{4}{3}$, find the ratio $\dfrac{2x+3y}{x-2y}$.

18. If $4x - 2y - 7z = 0$ and $3x + 8y - 29z = 0$, prove that $x = 3z$ and find the ratio $y : z$.

19. If $3x = y + 2z$ and $4z = 2x + y$, find the value of $\dfrac{x^2+z^2}{xy}$.

20. If $3x - 2y + 4z = x + 2y - 3z = 0$, find the ratios $x : y : z$.

21. The sides of an isosceles triangle are in the ratio $p:q:p$, and the perimeter is $2s$ in.; find the length of the base.

22. If $(a+b):(b+c):(c+a)=17:18:25$, prove that the triangle whose sides are of lengths a, b, c is right-angled.

Proportion. If $\frac{a}{b}$ and $\frac{c}{d}$ are equal ratios, the numbers a, b, c, d are said to be **in proportion**; and d is called the *fourth proportional* to a, b, c.

If $\frac{a}{b}=\frac{b}{c}$, c is called the *third proportional* to a, b; and b is called the *mean proportional* between a and c, therefore the mean proportional between two numbers is their geometric mean.

If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=\dots$, the numbers a, b, c, d, e, ... are said to be in **continued proportion**; therefore numbers in continued proportion are successive terms of a G.P., and may be expressed in the form a, ar, ar^2, ar^3, ...

Equal Ratios. If $\frac{a}{b}=\frac{c}{d}$, then $ad=bc$; hence we have also $\frac{a}{c}=\frac{b}{d}$.

Again, if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{b}+1=\frac{c}{d}+1$;

$$\therefore\ \frac{a+b}{b}=\frac{c+d}{d};$$

also $$\frac{b}{a}=\frac{d}{c};\quad \therefore\ \frac{b}{a}+1=\frac{d}{c}+1;$$

$$\therefore\ \frac{a+b}{a}=\frac{c+d}{c}.$$

These results have important geometrical interpretations.

If in Fig. 6, PQ is parallel to YZ, we know that $\frac{\mathsf{XP}}{\mathsf{PY}}=\frac{\mathsf{XQ}}{\mathsf{QZ}}$.

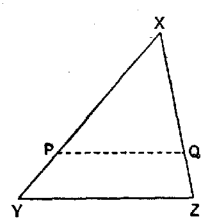


FIG. 6.

Hence we have $\frac{XP+PY}{PY}=\frac{XQ+QZ}{QZ}$, *that is*, $\frac{XY}{PY}=\frac{XZ}{QZ}$;

also $$\frac{XP+PY}{XP}=\frac{XQ+QZ}{XQ}, \text{ that is, } \frac{XY}{XP}=\frac{XZ}{XQ}.$$

These results are all special cases of the general statement that if $\frac{a}{b}$ and $\frac{c}{d}$ are equal ratios, any expression involving a and b, which can be expressed as a function of $\frac{a}{b}$ only, is equal to the corresponding expression involving c and d.

Thus $$\frac{a+b}{a-b}=\frac{\frac{a}{b}+1}{\frac{a}{b}-1}=\frac{\frac{c}{d}+1}{\frac{c}{d}-1}=\frac{c+d}{c-d}.$$

Proofs of such results are often put in the following form:

If $\frac{a}{b}=\frac{c}{d}$, let each ratio $=k$.

Then $\frac{a}{b}=k$ gives $a=bk$; $\frac{c}{d}=k$ gives $c=dk$.

$$\therefore \frac{a+b}{a-b}=\frac{bk+b}{bk-b}=\frac{b(k+1)}{b(k-1)}=\frac{k+1}{k-1};$$

and similarly $$\frac{c+d}{c-d}=\frac{dk+d}{dk-d}=\frac{k+1}{k-1};$$

$$\therefore \frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

The chief property of equal ratios is as follows:

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\dots$, then each ratio $=\frac{pa+qc+re+\dots}{pb+qd+rf+\dots}$.

Let each ratio $=k$.

$$\therefore a=bk,\ c=dk,\ e=fk,\ \dots .$$

$$\therefore \frac{pa+qc+re+\dots}{pb+qd+rf+\dots}=\frac{pbk+qdk+rfk+\dots}{pb+qd+rf+\dots}$$
$$=\frac{k(pb+qd+rf+\dots)}{pb+qd+rf+\dots}$$
$$=k.$$

Example 3. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that

$$\frac{2a-3c}{2b-3d}=\sqrt{\left(\frac{a^2+5c^2-3ae}{b^2+5d^2-3bf}\right)}.$$

Let each of the ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ equal k.

$$\therefore\ a=bk, \quad c=dk, \quad e=fk;$$

$$\therefore\ \text{left side}=\frac{2bk-3dk}{2b-3d}=\frac{k(2b-3d)}{2b-3d}=k;$$

and $$\text{right side}=\sqrt{\left(\frac{b^2k^2+5d^2k^2-3bfk^2}{b^2+5d^2-3bf}\right)}$$

$$=\sqrt{(k^2)}=k=\text{left side}.$$

EXERCISE VI. b

1. If $\frac{a}{b}$ and $\frac{c}{d}$ are unequal ratios, and if a, b, c, d are positive, it can be proved that $\frac{a+c}{b+d}$ lies *between* $\frac{a}{b}$ and $\frac{c}{d}$. Verify this statement for the ratios $\frac{3}{5}$ and $\frac{3}{4}$.

2. If $a=8$, $b=12$, $c=10$, $d=15$, write down the values of

(i) $\frac{a}{b}$ and $\frac{c}{d}$; (ii) $\frac{a+b}{a-b}$ and $\frac{c+d}{c-d}$; (iii) $\frac{a+b}{c+d}$;

(iv) $\frac{ac}{bd}$; (v) $\frac{a^2+c^2}{b^2+d^2}$; (vi) $\frac{7a^3-c^3}{7b^3-d^3}$.

3. Express $\frac{5a-7b}{2a+3b}$ as a function of $\frac{a}{b}$ only. If also $\frac{a}{b}=\frac{c}{d}$, express it in terms of c and d.

4. If $\frac{a}{b}=\frac{3}{5}$, find the values of (i) $\frac{a+b}{b}$; (ii) $\frac{a-b}{a}$; (iii) $\frac{a+3b}{3a-b}$.

5. Find the value of $\frac{x-y}{x+y}$ if (i) $\frac{x}{y}=\frac{7}{3}$; (ii) $\frac{x}{9}=\frac{y}{7}$; (iii) $11x=5y$.

6. Find numerical values for x, y, z if

$$\frac{a}{10}=\frac{b}{9}=\frac{a-b}{x}=\frac{a+b}{y}=\frac{3a-2b}{z}.$$

7. Express $\dfrac{2a^2-3ab+b^2}{a^2+5b^2}$ as a function of $\dfrac{a}{b}$ only. What is its value if (i) $\dfrac{a}{b}=2$; (ii) $\dfrac{a}{5}=\dfrac{b}{2}$? If $\dfrac{a}{b}=\dfrac{c}{d}$, express the fraction in terms of c and d.

8. Find numerical values for x, y, if

$$\frac{a}{6}=\frac{b}{8}=\frac{c}{9}=\frac{a+b-c}{x}=\frac{3a-b-c}{y}.$$

Also write down the value of $\dfrac{a-b}{b-c}$.

9. If $6x=9y=15z$, find the 3 smallest integers to which x, y, z are proportional.

10. Find the fourth proportional to

(i) 10, 15, 24; (ii) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$; (iii) ab, bc, ca.

11. Find the third proportional to

(i) 8, 12; (ii) $\frac{1}{4}$, $\frac{1}{6}$; (iii) a, $\dfrac{1}{a}$.

12. Find the mean proportional between

(i) 8, 18; (ii) a^3, ab^2.

13. Complete the relations,

$$\frac{x+y}{15}=\frac{x-y}{7}=\frac{2x}{}=\frac{2y}{}.$$

14. Complete the relations,

$$\frac{x+y+z}{20}=\frac{x+y-z}{14}=\frac{x-y+z}{11}=\frac{2x}{}=\frac{2y}{}=\frac{2z}{};$$

then write down the proportion, $x:y:z$.

15. If $\dfrac{x+y}{20}=\dfrac{x-y+z}{11}=\dfrac{x+y-z}{5}$, find $x:y:z$.

16. If $\dfrac{a}{b}=\dfrac{c}{d}$, prove that $\dfrac{3a-c}{3b-d}=\dfrac{a+5c}{b+5d}$.

17. If $\dfrac{a}{b}=\dfrac{c}{d}$, prove that $\dfrac{a^2}{a^2-b^2}=\dfrac{c^2}{c^2-d^2}$.

18. If $\dfrac{a}{b}=\dfrac{c}{d}$, prove that $\dfrac{a+b-c-d}{b-d}=\dfrac{a+b}{b}$.

19. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, prove that $\dfrac{a^3-c^3}{b^3-d^3}=\left(\dfrac{c+e}{d+f}\right)^3$.

20. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, complete the following relations,

(i) $\frac{a}{b}=\frac{a-c+e}{}=\frac{}{f-b-d}=\frac{ax-ey}{}$;

(ii) $\frac{a^2}{b^2}=\frac{ce}{}=\frac{}{(f-b)(f-d)}$.

21. If $\frac{a}{b}=\frac{b}{c}$, prove that $\frac{a^2+b^2}{(a+c)^2}=\frac{b^2}{b^2+c^2}$.

22. If $\frac{a}{b}=\frac{b}{c}$, prove that $\frac{a^2+ab+b^2}{a}=\frac{b^2+bc+c^2}{c}$.

23. If $\frac{a}{b}=\frac{b}{c}$, prove that $\frac{a-b}{b-c}=\sqrt{\left(\frac{a}{c}\right)}$.

24. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $\frac{ab+bc+ca}{bc+cd+db}=\left(\frac{a}{d}\right)^{\frac{2}{3}}$.

25. If $a : b : c = pq : p^2 : qr$, express $p : q : r$ in terms of a, b, c.

[*For further practice, see Appendix, Ex. F.P.* 6, *p.* 158. *For a revision exercise on Ch. V-VI, see Appendix, Ex. W.* 3, *p.* 149.]

CHAPTER VII

GRAPHICAL FORMS

Graphs of Related Functions

GRAPHS of some of the simpler algebraic functions have been drawn so frequently that their forms and shapes are now familiar; but it is useful to summarise the chief features. The following exercise is therefore intended for class-discussion, and it is suggested that *sketches* of groups of allied graphs given in the exercise should be shown on the blackboard.

Exercise VII. a

The Linear Function

1. *Without* making a table of values, draw rapidly on squared paper freehand graphs of

$$y=x,\ y=2x,\ y=\tfrac{1}{3}x,\ y=-x,\ y=-3x,\ y=-\tfrac{1}{3}x.$$

(a) What property is common to all of these graphs?

(b) What function is represented by a straight line through the origin with slope $\frac{1}{2}$? with slope -2?

(c) What is the graph of $y=mx$ where m is any given constant?

2. Draw on squared paper a freehand graph of $y=2x$. *Without* making a table of values, draw in the same figure the graph of $y=2x+1$. Then add the graphs of

$$y=2x-1,\ y=2x+3,\ y=2x-4.$$

(a) What property is common to all of these graphs?

(b) How far from 0 do these graphs cut the y-axis?

3. *Without* making a table of values, draw on squared paper freehand graphs of the following:

(i) $y=\frac{1}{2}x+2$ and $y=\frac{1}{2}x-3$;

(ii) $y=-\frac{4}{3}x+3$ and $y=-\frac{4}{3}x-2$.

4. State in words the method for drawing the graph of $y=mx+c$ where m and c are any given constants.

If the graph is given, how can you tell at a glance (i) whether m is + or −, (ii) whether c is + or − ?

[Lines may be drawn on a squared blackboard to represent given graphs and the following questions asked: (a) What is the slope? (b) Through what distance, *measured parallel to Oy*, must

the line be moved to make it pass through the origin? (c) What function of x does it represent ?]

The Quadratic Function

5. Draw carefully on squared paper the graph of $y=x^2$ for values of x from -3 to $+3$, with the origin in the middle of the paper.

Use this graph to draw, without making any further table of values, freehand graphs of

$$y=2x^2,\ y=\tfrac{1}{2}x^2,\ y=-x^2,\ y=-\tfrac{1}{2}x^2,\ y=-2x^2.$$

(a) Which of these graphs have a lowest point? "*Head down.*"

(b) Which of these graphs have a highest point? "*Head up.*"

6. Draw carefully on squared paper the graph of $y=2x^2$ for values of x from -3 to $+3$. Use this graph to draw rapidly freehand graphs of $y=2x^2+5$, $y=2x^2-10$; are they "head up" or "head down"?

7. Show in one table the values of x^2, $(x+2)^2$, $(x-1)^2$ for $x=4, 3, 2, 1, 0, -1, -2, -3, -4$.

Draw carefully the graph of $y=x^2$ from $x=4$ to $x=-4$ and then cut it out.

(a) In what position must you hold it to show the graph of

(i) $y=(x+2)^2$; (ii) $y=(x-1)^2$?

(b) How must you hold it to show the graph of

(i) $y=-x^2$; (ii) $y=-(x+1)^2$; (iii) $y=-(x-2)^2$?

(c) How must you hold it to show the graph of

(i) $y=(x+2)^2+5$; (ii) $y=(x+2)^2-10$;

(iii) $y=-(x+1)^2+10$; (iv) $y=-(x+1)^2-5$?

8. Fig. 7 shows the graph of $y=x^2$ in various positions. What functions of x do these graphs represent?

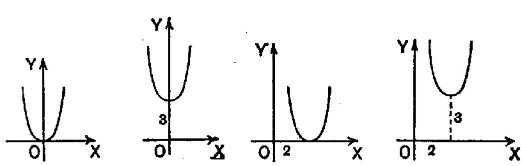


FIG. 7.

9. Hold the book upside down and suppose the arrows showing the directions of OX, OY in Fig. 7 to be reversed. What functions of x do these graphs now represent?

10. Any quadratic function can be expressed in the form $a(x+b)^2+c$, where a, b, c are constants.

(i) What is the condition that its graph is "head up"? "head down"?

(ii) How can you find the position of the "head" of the graph?

11. By making a table of values, discover the connection between the graphs of $y=x(x-1)$ and $y=(x-3)(x-4)$.

12. Draw the graphs of $y=x^2+2x+3$ and $y=x^2-2x-1$. Verify from the drawing that the two graphs are exactly equal curves, and prove algebraically that each is an equal curve to the graph of $y=x^2$.

Other Functions

13. Draw carefully the graph of $y=x^3$ for values of x from -3 to 3. Sketch in the same figure the graphs of $y=\frac{1}{2}x^3$ and $y=-x^3$.

14. Without making a table of values, sketch the graphs of $y=x^4$ and $y=x^4+10$.

15. What is the connection between the following pairs of graphs?

(i) $y=x^2$; $x=y^2$.	(ii) $y=x^2$; $y=\pm\sqrt{x}$.
(iii) $y=x^3$; $x=y^3$.	(iv) $y=x^3$; $y=\sqrt[3]{x}$.
(v) $y=10^x$; $x=10^y$.	(vi) $y=10^x$; $y=\log x$.

Graphs with Missing Points

If we are given the formula

$$xy=6,$$

we can make a table of values showing the values of y for selected values of x.

Thus, if $x=3$, $y=2$; if $x=\frac{1}{10}$, $y=60$; if $x=-2$, $y=-3$; if $x=-\frac{1}{5}$, $y=-30$; etc.

But it is impossible to suppose that $x=0$, because $0\times y=0$ and so cannot equal 6.

There is a **break** *in the table of values at* $x=0$.

If $xy=6$, $y=\frac{6}{x}$ unless $x=0$; if $xy=6$, we know that x cannot equal 0; therefore there is no such number as $\frac{6}{0}$; the expression $\frac{6}{0}$ has *no meaning*.

The table of values for the formula, $xy=6$, may be arranged as follows :

x	-4	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{10}$	0	$\frac{1}{10}$	$\frac{1}{2}$	1	2	3	4
$y=\frac{6}{x}$	$-1{\cdot}5$	-2	-3	-6	-12	-60	*	60	12	6	3	2	$1{\cdot}5$

↑ Missing value.

The symbol * is inserted in the table where $x=0$ to show that there is no value for y or $\frac{6}{x}$ when $x=0$; it marks the place where there is a missing point or break in the graph. **Points on one side of the break cannot be joined to points on the other side of the break,** when the graph is drawn.

Fig. 8 shows points obtained by plotting some of the values in the table, given above, and the corresponding portion of the

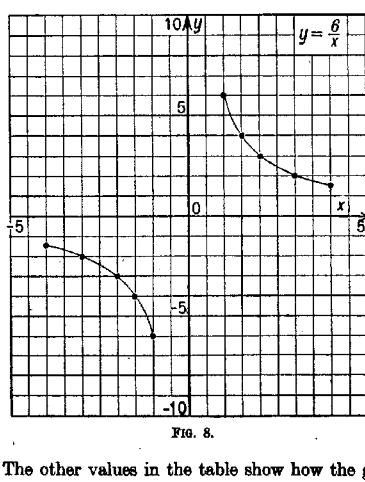


FIG. 8.

graph. The other values in the table show how the graph runs for small positive values of x, that is, to the right of the break, and for small negative values of x, that is, to the left of the break.

The reader should now draw the graph of $y=\frac{6}{x}$ on a large sheet of paper and show as much of the graph as the size of the paper allows.

If $xy=6$, we can give y any value *except* $y=0$; it is impossible to suppose that $y=0$, because $x\times 0=0$ and so cannot equal 6; this means that if we make a table of values showing the values of x for selected values of y, there will be a missing value in the table at $y=0$, and a corresponding break in the graph. Points above the x-axis, where y is positive, cannot be joined to points below the x-axis, where y is negative, when the graph is drawn; see Fig. 8.

Example 1. Represent graphically the formula,

$$(y-1)(x-2)=3,$$

and solve the simultaneous equations,

$$(y-1)(x-2)=3,\ 8x-10y=15.$$

From the formula, $(y-1)(x-2)=3$, we can calculate values of y for any selected values of x, *except* $x=2$. If $x=2$, $(y-1)(x-2)=0$ and so cannot equal 3. There is a *break* in the graph at $x=2$.

Before obtaining a table of values, make y the subject of the formula.

Divide each side by $x-2$, $\therefore\ y-1=\frac{3}{x-2}$;

$$\therefore\ y=\frac{3}{x-2}+1.$$

x	-4	-2	-1	0	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4	5	6
$x-2$	-6	-4	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3	4
$\frac{3}{x-2}$	$-0{\cdot}5$	$-0{\cdot}75$	-1	$-1{\cdot}5$	-3	-6	*	6	3	$1{\cdot}5$	1	0.75
$y=\frac{3}{x-2}+1$	$0{\cdot}5$	$0{\cdot}25$	0	$-0{\cdot}5$	-2	-5	*	7	4	$2{\cdot}5$	2	1.75

Missing value.

Fig. 9 shows points obtained by plotting the values in this table and the corresponding part of the graph.

If $y=1$, $(y-1)(x-2)=0$ and so cannot equal 3. This means that there is no value of x for which $y=1$ and that there is a

corresponding break in the graph where $y=1$. When the graph is drawn, points for which y is greater than 1 cannot be joined to points where y is less than 1.

For the graph of $8x-10y=15$, we have $10y=8x-15$;

$$\therefore\ y=\frac{8x-15}{10}.$$

When $x=0$, $y=-\frac{15}{10}=-1\cdot5$; when $x=5$, $y=\frac{40-15}{10}=2\cdot5$.

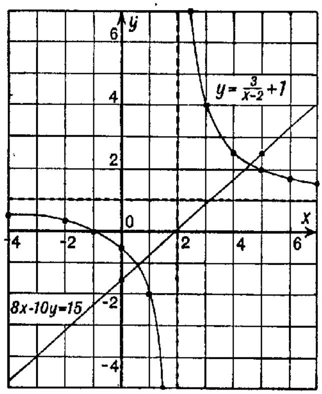


FIG. 9.

Fig. 9 shows the graph of $y=\frac{8x-15}{10}$, obtained by joining these points. The graphs intersect at the points,

$x=4\cdot6$, $y=2\cdot2$ and $x=0\cdot55$, $y=-1\cdot05$ approximately.

The reader should compare these solutions with those obtained by calculation.

Since $\frac{3}{x-2}+1=\frac{3+x-2}{x-2}=\frac{x+1}{x-2}$, the curved graph in Fig. 9 represents the function, $y=\frac{x+1}{x-2}$.

From these examples, it is clear that the graph of any fractional function has a gap or break for any value of x for which the denominator of the function is zero.

EXERCISE VII. b

1. Make a table of values from $x = -3$ to $x = +3$ for each of the following functions, and show where there are missing values.

(i) $y = \frac{12}{x}$; (ii) $y = \frac{12}{x-1}$; (iii) $y = \frac{12}{x+2}$;

(iv) $y = \frac{x}{x+1}$; (v) $y = \frac{x-2}{x+3}$; (vi) $y = 2 - \frac{1}{x-3}$.

2. Make a table of values from $x = -3$ to $x = +3$ for each of the following formulae, and show where breaks occur in the corresponding graphs. Sketch rough graphs on plain paper.

(i) $xy = 60$; (ii) $y(x-2) = 60$; (iii) $x(y-1) = 60$;
(iv) $(x-1)(y+2) = 60$; (v) $(x+3)(y-2) = 60$.

3. Find some value of x such that (i) $\frac{1}{x} > 100$; (ii) $\frac{1}{x} < -100$ (iii) $\frac{1}{x-2} > 100$; (iv) $\frac{1}{x-2} < -100$.

4. Draw as much as your paper allows of the graph of $y = \frac{60}{x}$, taking the origin in the middle of the paper, and 1 inch on each axis to represent 10 units.

5. Repeat No. 4 for the graph of $y = \frac{60}{x-10} - 20$.

6. Repeat No. 4 for the graph of $y = \frac{60}{x+20} + 10$.

7. Draw, for values of x from -1 to 2, as much of the graph of $4xy = 3$ as your paper allows. Solve graphically the simultaneous equations, $4xy = 3$, $y = 3(1 - \frac{1}{2}x)$. Check by calculation.

8. Draw, for values of x from -1 to 2, as much of the graph of $y = 1 + \frac{1}{x}$ as your paper allows. Solve graphically the simultaneous equations, $y = 1 + \frac{1}{x}$, $y = 3x - 1$. Check by calculation.

9. Draw for values of x from -5 to $+5$ as much of the graphs of $y = 2 + \frac{1}{x}$ and $y = \frac{x^2}{10}$ as your paper allows.

What cubic equation in x can be solved from these graphs, and what are its roots?

10. Draw, for values of x from -3 to $+3$, as much of the graph of $y = \frac{1}{x^2}$ as your paper allows. Solve graphically the simultaneous equations, (i) $y = \frac{1}{x^2}$, $x + y = 2$; (ii) $y = \frac{1}{x^2}$, $y = 2x + 1$.

What cubic equations in x can be solved from these graphs, and what are the roots?

11. Draw, for values of x from -5 to $+3$, as much of the graph of $y=\frac{1-x}{2+x}$ as your paper allows. Solve the simultaneous equations, $y(x+2)+x=1$, $y=2x$. Check by calculation.

12. Draw for values of x from -3 to $+3$, as much of the graphs of $y=\frac{1}{x}$ and $y=x^2-4$ as your paper allows.

What cubic equation in x can be solved from these graphs, and what are the roots ?

13. Solve graphically the simultaneous equations,

$$(x-2)(y-4)=15, \quad 2y-3x=12.$$

Check by calculation.

14. Solve graphically the simultaneous equations,

$$y=x^2, \quad y=\frac{2(x+3)}{x-3}.$$

Consider values of x from -3 to $+5$.
What cubic equation in x can be solved from these graphs, and what are the roots ?

15. Draw, for values of x from -7 to $+7$, as much as your paper allows of the graphs of $y=x+\frac{9}{x}$ and $y=\frac{3}{8}(x^2-25)$.

What cubic equation in x can be solved from these graphs, and what are the roots ?

[*For additional examples, see Appendix, Ex. T. 6, p.* 174.]

Empirical Formulae

If we obtain by measurement or observation a table of values of a variable y for selected values of x, we can show the relation between y and x by a graph. If the plotted points lie on a straight line, or nearly so, the relation is of the form $y=a+bx$, and methods for finding the values of a and b have been explained in Part II, Ch. X, see Example 5, p. 169; the position of the "best-fit" line should be determined by using a piece of black thread (see p. 29).

It has also been shown in Ch. II, see p. 30, how to test graphically whether y varies as any *given* power of x. If there is reason to expect a relation of the form $y=ax^n$, although the value of n is unknown, the following process is used.

If $y=ax^n$, then $\log y=\log(ax^n)=\log a+\log(x^n)$;

$$\therefore \log y=\log a+n\log x.$$

Consequently if values of $\log y$ are plotted against values of $\log x$, the points should lie on a straight line, or nearly so; and then the position of the best-fit line determines the values of $\log a$ and n, as before.

In numerical work, it may be assumed, unless otherwise stated, that the logarithmic base is 10.

Example 2. The relation, $\log y = 0{\cdot}74 + 2{\cdot}3 \log x$, has been obtained from a straight-line graph, representing values of $\log y$ plotted against values of $\log x$. Find y in terms of x.

$$\log y = 0{\cdot}74 + 2{\cdot}3 \log x = \log 5{\cdot}5 + \log(x^{2\cdot3})$$
$$= \log(5{\cdot}5x^{2\cdot3});$$
$$\therefore\ \mathbf{y = 5{\cdot}5x^{2\cdot3}}.$$

Example 3. The relation, $\log y = 0{\cdot}85 + 0{\cdot}58x$, has been obtained from a straight-line graph, representing values of $\log y$ plotted against values of x. Find y in terms of x.

$$\log y = 0{\cdot}85 + 0{\cdot}58x;\ \therefore\ y = 10^{0\cdot85 + 0\cdot58x};$$
$$\therefore\ y = 10^{0\cdot85} \times 10^{0\cdot58x} = 7{\cdot}1 \times 10^{0\cdot58x}.$$

Since $10^{0\cdot58} \simeq 3{\cdot}8$, this relation may also be written, $y = 7{\cdot}1(3{\cdot}8)^x$.

EXERCISE VII. c

1. If corresponding values of x and y are given by the following table, find the simplest expression for y in terms of x.

x - -	1	3	4	6
y - -	1	5	7	11

2. Repeat No. 1 for the following table:

x - -	0	1	3	6
y - -	8	5	−1	−10

3. The following values of the effort P lb. required to raise a load of W lb. by means of a differential pulley have been found by experiment.

W - -	100	140	180	220	260	300
P - -	18·2	24·7	30·4	36·8	42·2	48·6

Plot values of P against values of W and, allowing for small experimental errors, obtain a relation expressing P in terms of W.

4. The following table shows the current C amperes which will fuse a copper wire of diameter d mm.:

d	0·2	0·4	0·5	0·6	0·8
C	7·2	20	28	37	57

Plot values of log C against values of log d, and use the graph to express C in terms of d.

5. The following table shows, with small errors, values of y corresponding to selected values of x.

x	2	3	3·5	4	4·5
y	2	6·7	10·7	16	22·8

Plot values of log y against values of log x and use the graph to express y in terms of x.

6. The following readings connect the candle-power C and the voltage V of an incandescent lamp:

C	20·68	23·24	26·00	28·96
V	94	98	102	106

Plot values of log C against values of log V, and use the graph to express C in terms of V.

7. The following table shows corresponding values of x and y:

x	2	2·5	3	3·5	4
y	7·58	10·8	14·5	18·6	23·0

Test graphically whether x and y are connected by a law of the form $y=ax^n$, and if so find a and n.

8. The following table shows, with small errors, values of y corresponding to selected values of x:

x	2	3	4	5	6	7	8
y	2·5	4·3	6·7	10	15	23·3	40

Plot values of $\frac{y}{x}$ against values of y, and use the graph to express x in terms of y.

9. The following table shows, with small errors, values of y corresponding to selected values of x:

x . .	1	2	4	6	8
y . .	14	18	36	66	120

Plot values of $\log y$ against values of x, and use the graph to find y in terms of x.

CHAPTER VIII

FURTHER PROCESSES

Long Multiplication and Division

Detached Coefficients. Examples of long multiplication and division were given in Part II, Ch. XI. These processes may now be revised and abbreviated.

Example 1. Multiply $2x^3+x^2+3$ by $3x^2+2x+1$, and compare the working with that of the corresponding example in Arithmetic when $x=10$.

If $x=10$, $2x^3+x^2+3=2000+100+3=2103$,

and $\qquad 3x^2+2x+1=\ 300+\ 20+1=\ 321.$

$$\begin{array}{llllll}
2x^3 & +x^2 & & +3 & & \\
3x^2 & +2x & +1 & & & \\
\hline
6x^5 & +3x^4 & & +9x^2 & & \\
 & 4x^4 & +2x^3 & & +6x & \\
 & & 2x^3 & +\ x^2 & & +3 \\
\hline
\mathbf{6x^5} & \mathbf{+7x^4} & \mathbf{+4x^3} & \mathbf{+10x^2} & \mathbf{+6x} & \mathbf{+3}
\end{array}
\qquad\qquad
\begin{array}{r}
2103 \\
321 \\
\hline
6309 \\
4206 \\
2103 \\
\hline
\mathbf{675063}
\end{array}$$

The chief difference between these two examples is the "carrying" from one column to the next in the arithmetical work, which cannot be done in algebra.

Expressions should always be arranged in ascending (or descending) powers of some letter. *If this is done and if a zero is inserted for each missing power*, the working of the above *algebraical* example can be abbreviated as follows:

$$\begin{array}{llllll}
2 & +1 & +0 & +3 & & \\
3 & +2 & +1 & & & \\
\hline
6 & +3 & +0 & +9 & & \\
 & 4 & +2 & +0 & +6 & \\
 & & 2 & +1 & +0 & +3 \\
\hline
\mathbf{6x^5} & \mathbf{+7x^4} & \mathbf{+4x^3} & \mathbf{+10x^2} & \mathbf{+6x} & \mathbf{+3}
\end{array}$$

This is called the **method of detached coefficients.** *The only object* of the method is to save time.

Example 2. Multiply $3a^3 - 2ab^2 + 5b^3$ by $6a^2 - 7ab + 4b^2$.

These expressions are arranged in descending powers of a; we can use "detached coefficients" because each expression is *homogeneous*. In $3a^3 - 2ab^2 + 5b^3$, each term is of order 3, $a^3 = aaa$, $ab^2 = abb$, $b^3 = bbb$; in $6a^2 - 7ab + 4b^2$, each term is of order 2.

Working in full.

$$
\begin{array}{llllll}
3a^3 & & -\ 2ab^2 + & 5b^3 & & \\
\underline{6a^2 - 7ab} & \underline{+\ 4b^2} & & & & \\
\hline
18a^5 & & -\ 12a^3b^2 & +\ 30a^2b^3 & & \\
 & -21a^4b & & +\ 14a^2b^3 & -\ 35ab^4 & \\
 & & 12a^3b^2 & & -\ 8ab^4 & +\ 20b^5 \\
\hline
18a^5 & -\ 21a^4b & & +\ 44a^2b^3 & -\ 43ab^4 & +\ 20b^5 \\
\hline
\end{array}
$$

Detached Coefficients.

$$
\begin{array}{rrrrrr}
3 & +\ 0 & -\ 2 & +\ 5 & & \\
6 & -\ 7 & +\ 4 & & & \\
\hline
18 & +\ 0 & -\ 12 & +\ 30 & & \\
 & -\ 21 & +\ 0 & +\ 14 & -\ 35 & \\
 & & 12 & +\ 0 & -\ 8 & +\ 20 \\
\hline
18 & -\ 21 & +\ 0 & +\ 44 & -\ 43 & +\ 20 \\
\hline
\end{array}
$$

$\therefore$ the product is $\mathbf{18a^5 - 21a^4b + 44a^2b^3 - 43ab^4 + 20b^5}$.

Example 3. Find the coefficient of x^3 in

$$(1 + 2x - 3x^2 - x^3)(5 - x - 2x^2 + 4x^3).$$

Link together the pairs of terms whose product involves x^3.
These are $1 \times 4x^3$, $2x \times (-2x^2)$, $(-3x^2)(-x)$, $(-x^3) \times 5$.
$\therefore$ the coefficient of x^3 is $4 - 4 + 3 - 5 = -2$.

Example 4. Divide $x^3 - 11x + 2$ by $x - 3$.

Working in full.

$$
\begin{array}{r|l}
 & x^2 + 3x - 2 \\
\hline
x - 3 & x^3 \qquad - 11x + 2 \\
 & \underline{x^3 - 3x^2} \\
 & 3x^2 - 11x \\
 & \underline{3x^2 - 9x} \\
 & -\ 2x + 2 \\
 & \underline{-\ 2x + 6} \\
 & -4
\end{array}
$$

Detached Coefficients.

$$
\begin{array}{r|l}
 & 1 + 3 - 2 \\
\hline
1 - 3 & 1 + 0 - 11 + 2 \\
 & \underline{1 - 3} \\
 & 3 - 11 \\
 & \underline{3 - 9} \\
 & -\ 2 + 2 \\
 & \underline{-\ 2 + 6} \\
 & -4
\end{array}
$$

The quotient is $\mathbf{x^2 + 3x - 2}$, the remainder is -4.

Example 5. Divide $16x^4 - 16x^3 + 11x - 12$ by $4x^2 - 3$.

$$
\begin{array}{r|rrrrr}
 & 4 & -\ 4 & +\ 3 & & \\
\hline
4+0-3 & 16 & -16 & +\ 0 & +11 & -12 \\
 & 16 & +\ 0 & -12 & & \\
\hline
 & & -16 & +12 & +11 & \\
 & & -16 & +\ 0 & +12 & \\
\hline
 & & & 12 & -\ 1 & -12 \\
 & & & 12 & +\ 0 & -\ 9 \\
\hline
 & & & & -\ 1 & -\ 3
\end{array}
$$

The quotient is $\mathbf{4x^2 - 4x + 3}$, the remainder is $\mathbf{-x - 3}$.

Example 6. Divide $x^3 - y^3$ by $x - y$.

We can regard $x^3 - y^3$ as $x^3 + 0 \,.\, x^2y + 0 \,.\, xy^2 - y^3$, and then use detached coefficients.

Working in full.

$$
\begin{array}{r|llll}
 & x^2 & +xy & +y^2 & \\
\hline
x-y & x^3 & & & -y^3 \\
 & x^3 & -x^2y & & \\
\hline
 & & x^2y & & \\
 & & x^2y & -xy^2 & \\
\hline
 & & & xy^2 & -y^3 \\
 & & & xy^2 & -y^3 \\
\hline
\end{array}
$$

Detached Coefficients.

$$
\begin{array}{r|rrrr}
 & 1 & +1 & +1 & \\
\hline
1-1 & 1 & +0 & +0 & -1 \\
 & 1 & -1 & & \\
\hline
 & & 1 & +0 & \\
 & & 1 & -1 & \\
\hline
 & & & 1 & -1 \\
 & & & 1 & -1 \\
\hline
\end{array}
$$

The quotient is $\mathbf{x^2 + xy + y^2}$; no remainder.

EXERCISE VIII. a

Find the product of the following:

1. $x^2 - 2x + 3,\ 2x + 1.$
2. $a^2 - 2ab + b^2,\ a + b.$
3. $2x^2 + x - 1,\ x^2 - 2x + 4.$
4. $x^2 + xy + y^2,\ x - y.$
5. $a^2 + ab + b^2,\ a^2 - ab + b^2.$
6. $1 - 4x - 3x^2,\ 1 + 2x - 6x^2.$
7. $3x^2 - x - 2,\ 1 - x + 2x^2.$
8. $a^2 - ab + b^2,\ a + b.$
9. $2a^3 - a + 3,\ a^3 - 2a^2 + 3.$
10. $1 + 2y + 3y^2 + 4y^3,\ 1 - 2y + y^2.$
11. $a + b + c,\ a + b + c.$
12. $x + y + z,\ x - y + z.$

Divide:

13. $2x^3 - x^2 + x - 1$ by $x - 2$.
14. $3x^2 + 2x - 5$ by $x - 1$.
15. $6x^3 - 5x^2 - 8x + 3$ by $2x - 3$.
16. $9x^2 + 5x$ by $3x - 1$.
17. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.
18. $x^4 - 3xy^3 + 2y^4$ by $x - y$.
19. $y^4 - z^4$ by $y + z$.
20. $8 - 10x + 5x^2 - x^3$ by $2 - x$.

21. $x^4+x^3-5x^2-3x+2$ by x^2+2x-1.

22. $x^4+x^2y^2+y^4$ by x^2+xy+y^2.

23. x^4-x^3-2x+5 by x^2-x-2.

24. x^3+y^3 by x^2-xy+y^2. 25. a^6-b^6 by $a^4+a^2b^2+b^4$.

Pick out the following coefficients :

26. Coefficients of x^3 and x in $(3x^2-2x-1)(2x^2+3x-4)$.

27. Coefficients of x^4 and x^2 in $(x^3-3x^2+x-2)(2x^3+x^2-3x+1)$.

28. Coefficients of x^2 and x^3 in $(1+2x+3x^2+4x^3)(1-2x+x^2)$.

29. Coefficients of x and x^4 in $(2-x+3x^2-x^3+x^4)^2$.

30. Coefficients of a^3b and a^2b^2 in $(a^2+ab+b^2)(a^2-ab+b^2)$.

31. Expand in ascending powers of x as far as x^3 :

(i) $(2-x+3x^2-x^3-x^4)^2$; (ii) $(2-x+3x^2-x^3-x^4)^3$;
(iii) $(1+x)^2(2-x)(1+3x-x^2)$.

32. Expand (i) $(a+b)^3$; (ii) $(a-b)^3$.

33. Multiply $3x-2+\dfrac{1}{x}$ by $x+1-\dfrac{2}{x}$.

34. Prove that $\dfrac{x^2}{x-2}=x+2+\dfrac{4}{x-2}$; and express as a similar mixed fraction $\dfrac{x^3}{x+2}$.

35. What is the constant term in $\left(x^2-2x-3+\dfrac{1}{x}+\dfrac{3}{x^2}\right)^2$?

36. By using long division, prove that

$$\frac{1}{1+x}=1-x+x^2-x^3+\frac{x^4}{1+x}.$$

37. Find the first 3 terms and the last 3 terms in the quotient when $x^{11}-11x+10$ is divided by x^2-2x+1.

38. Using detached coefficients, find the first 3 terms in the expansion of $(1+x)^6$.

The Remainder Theorem

Find the remainder when ax^2+bx+c is divided by $x-k$.

$$\begin{array}{r|l}
 & ax+(ak+b) \\ \hline
x-k & ax^2+bx+c \\
 & \underline{ax^2-akx} \\
 & \quad (ak+b)x+c \\
 & \quad \underline{(ak+b)x-k(ak+b)} \\
 & \qquad\qquad \underline{\underline{ak^2+bk+c}}
\end{array}$$

This work shows that the remainder is equal to the result of putting $x=k$ in the dividend. The reader should now prove by actual division that this is also true if the dividend is

$$ax^3+bx^2+cx+d;$$

and the same method may be used for any polynomial,

$$ax^n+bx^{n-1}+cx^{n-2}+\dots.$$

The result can, however, be established in another way, as follows:

The long division process in Example 4 proves that

$$x^3-11x+2\equiv(x-3)(x^2+3x-2)-4.$$

Since this is an identity, it is true for any value of x we like to substitute. If we put $x=3$, the *right* side becomes

$$0\times(9+9-2)-4=0-4=-4;$$

this is the *remainder* in the division sum. Therefore if we put $x=3$ in the *left* side, the result must equal the remainder, when the expression on the left side is divided by $(x-3)$.

This argument (cf. Example 7 below) holds whenever the divisor is of the first degree.

Thus if any polynomial in x

$$a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$$

is divided by $x-k$, then the **remainder** equals

$$a_nk^n+a_{n-1}k^{n-1}+\dots+a_1k+a_0.$$

And, in particular, $x-k$ *is a factor* of the polynomial, if

$$a_nk^n+a_{n-1}k^{n-1}+\dots+a_1k+a_0=0.$$

Example 7. What is the remainder if $2x^3+x^2+3$ is divided by $(x+2)$?

Without working through the long division sum, we can say

$$2x^3+x^2+3\equiv(x+2)(\text{quotient})+\text{remainder}\dots\dots\dots\dots\dots\dots\text{(i)}$$

where the remainder does not contain x.

Since this is an identity, it is true for any value of x.

Put $x=-2$, because this makes $x+2$ zero.

$$\therefore\ 2(-2)^3+(-2)^2+3=0\times(\dots\dots)+\text{remainder}\dots\dots\dots\dots\dots\text{(ii)}$$

$$\therefore\ -16+4+3=0+\text{remainder};$$

$$\therefore\ \text{the remainder}=-9.$$

Check:

$$\begin{array}{r|l} & 2x^2 - 3x + 6 \\ \hline x+2 & 2x^3 + x^2 \qquad + 3 \\ & \underline{2x^3 + 4x^2} \\ & -3x^2 \\ & \underline{-3x^2 - 6x} \\ & \qquad 6x + 3 \\ & \qquad \underline{6x + 12} \\ & \qquad\quad -9 \end{array}$$

Quotient $2x^2 - 3x + 6$.
Remainder -9.

$\therefore$ equation (i) written in full is

$$2x^3 + x^2 + 3 \equiv (x+2)(2x^2 - 3x + 6) - 9,$$

and equation (ii) written in full is

$$2(-2)^3 + (-2)^2 + 3 = 0 \times [2(-2)^2 - 3(-2) + 6] - 9 = 0 - 9.$$

But if we only want to find the remainder, there is no need to fill in what the quotient is.

Example 8. Prove that $x - 2$ is a factor of $x^3 - x^2 - 5x + 6$.

To obtain the remainder when $x^3 - x^2 - 5x + 6$ is divided by $x - 2$, put $x = 2$ in the expression. [This makes $x - 2$ zero.]

$\therefore$ the remainder $= 2^3 - 2^2 - 5(2) + 6 = 8 - 4 - 10 + 6 = 0$.

Since the remainder is zero, $x - 2$ is a factor.

Example 9. Find the factors of $x^3 + 2x^2 - 5x - 6$.

Try putting $x = 1, -1, 2, -2, 3, -3, 6, -6$, and see if the result is zero in any case. (These are the various factors of 6.)

If $x = 1$, expression is $1 + 2 - 5 - 6 = -8$.

If $x = -1$, expression is $-1 + 2 + 5 - 6 = 0$.

$\therefore$ $x + 1$ is a factor.

The other factor can now be found by long division.

Then $x^3 + 2x^2 - 5x - 6 = (x+1)(x^2 + x - 6)$
$= (\mathbf{x} + \mathbf{1})(\mathbf{x} + \mathbf{3})(\mathbf{x} - \mathbf{2})$.

Check this by putting $x = -3$ and $x = 2$; in each case the result should be zero.

After a little practice, quotients may be found by inspection. Thus, in the example above, we know that $x + 1$ is a factor of $x^3 + 2x^2 - 5x - 6$.

$\therefore$ the quotient must start with x^2 and must end with -6,

$$\therefore\ x^3 + 2x^2 - 5x - 6 = (x+1)(x^2 \ldots\ -6).$$

When we multiply up the right side, we obtain $1 \,.\, x^2$ from the terms already there; we need $2x^2$ in all; $\therefore$ the missing term is x.

Thus $$x^3+2x^2-5x-6=(x+1)(x^2+x-6).$$

Now *check* by picking out the coefficient of x on the right side, this is $-6+1=-5$.

Example 10. Factorise $(xy+yz+zx)(x+y+z)-xyz$.

Put $y=-z$ in the given expression; we then obtain

$$(-xz-z^2+zx)(x-z+z)+xz^2=(-z^2)(x)+xz^2=0;$$

$$\therefore\ (y+z) \text{ is a factor.}$$

Similarly we can prove that $(z+x)$ and $(x+y)$ are factors.

$\therefore$ the expression is divisible by $(y+z)(z+x)(x+y)$.

But the expression is a function of 3 dimensions; $\therefore$ any remaining factor must be numerical;

$$\therefore\ (xy+yz+zx)(x+y+z)-xyz\equiv k(y+z)(z+x)(x+y)$$

where k is a constant, that is, independent of x, y, z.

Put $x=0$, $y=1$, $z=1$;

then $(1)(2)-0=k(2)(1)(1)$; $\therefore\ 2k=2$; $\therefore\ k=1$.

$$\therefore \text{ the expression} =(\mathbf{y}+\mathbf{z})(\mathbf{z}+\mathbf{x})(\mathbf{x}+\mathbf{y}).$$

EXERCISE VIII. b

Find the remainder in the following cases:

1. $(x^2-5x+4)\div(x-1)$.
2. $(x^2+3x+2)\div(x+1)$.
3. $(3x^2+5x-2)\div(x+2)$.
4. $(3x^2-11x+6)\div(x-3)$.
5. $(2x^2+3x-2)\div(2x-1)$.
6. $(3x^2+x-2)\div(3x-2)$.
7. $(2x^2-x-3)\div(x-1)$.
8. $(3x^2+4x+1)\div(x+2)$.
9. $(x^3+7x^2+4x-2)\div(x+1)$.
10. $(x^3-2x^2+3x-6)\div(x-2)$.
11. $(2x^3-x^2-3x-1)\div(2x+1)$.
12. $(2x^3+5x^2-5x-6)\div(x+3)$.

13. Prove that $x-1$ is a factor of $3x^3-x^2-x-1$, and write down the other factor.

14. Prove that $x+1$ is a factor of $2x^3+3x^2-1$, and write down the other factors.

15. Prove that $x+2$ is a factor of $x^3-x^2-10x-8$, and write down the other factors.

16. Prove that $x-3$ is a factor of x^3-x^2-9x+9, and write down the other factors.

17. Prove that $x+1$ is a factor of x^3+1 and write down the other factor.

18. Prove that $x-a$ is a factor of x^3-a^3 and write down the other factor.

Find the factors of the following :

19. x^3-2x^2-5x+6.
20. x^3-4x^2+x+6.
21. $x^3-21x+20$.
22. $x^3+4x^2y-15xy^2-18y^3$.
23. $2a^3+a^2b-13ab^2+6b^3$.
24. $2+5x-28x^2-15x^3$.
25. For what value of a is $x-1$ a factor of x^3-7x+a ?
26. For what value of b is $x+1$ a factor of x^3+bx-5 ?
27. For what value of c is $x-1$ a factor of x^3+cx^2-5x+6 ? What are then the other factors ?
28. For what values of a, b are $x+1$ and $x-2$ factors of x^3+ax^2+2x+b ? What then is the other factor ?
29. For what values of b, c are $x+2$ and $2x-1$ factors of $4x^3+bx+c$? What then is the other factor ?
30. Prove that x^2+x-2 is a factor of x^4-x^3+4x-4.
31. Solve the equation $3x^3-7x^2+4=0$.
32. Solve the equation $x^3=7x+6$.
33. Solve the equation $6x^3-11x^2-37x+70=0$.
34. Solve the equation $4x^3-8x^2+x+3=0$.
35. Prove that $a-b$ is a factor of $a^2(b-c)+b^2(c-a)+c^2(a-b)$.
36. Prove that $a-b$ is a factor of $(b-c)^3+(c-a)^3$.
37. Prove that $x-y$ is a factor of

$$yz(y^2-z^2)+zx(z^2-x^2)+xy(x^2-y^2).$$

38. Prove that $x+y$ is a factor of

$$xy(x+y)+yz(y+z)+zx(z+x)+2xyz.$$

39. Find 3 factors of $x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)$.

[*For additional examples, see Appendix, Ex. T. 7, p.* 176.]

Further Factors and Fractions

The previous exercises have included the following identities :

$$A^3+B^3=(A+B)(A^2-AB+B^2) \quad \text{.........................(i)}$$

$$A^3-B^3=(A-B)(A^2+AB+B^2) \quad \text{.........................(ii)}$$

$$A^4+A^2B^2+B^4=(A^2+AB+B^2)(A^2-AB+B^2) \quad \text{............(iii)}$$

In (i) and (ii) the first factor is easy to remember ; the second factor may then be obtained by division.

(iii) may be factorised by expressing it as the difference of two squares;

$$A^4+A^2B^2+B^4=A^4+2A^2B^2+B^4-A^2B^2=(A^2+B^2)^2-A^2B^2$$
$$=(A^2+B^2+AB)(A^2+B^2-AB).$$

Perfect Square. The expression $(A+B+C)^2$ can be expanded by ordinary multiplication, or as follows:

$$(A+B+C)^2=[A+(B+C)]^2=A^2+2A(B+C)+(B+C)^2$$
$$=A^2+2AB+2AC+B^2+2BC+C^2$$
$$=\mathbf{A^2+B^2+C^2+2BC+2CA+2AB}.$$

In words, to obtain the square of the sum of three terms, add together the squares of each term and twice the product of every two of them.

The same rule holds for the square of the sum of any number of terms.

Perfect Cube. $(A+B)^3=(A+B).(A^2+2AB+B^2)$
$$=\mathbf{A^3+3A^2B+3AB^2+B^3}.$$

The expansion for $(A-B)^3$ may be obtained in the same way, *or* by writing $-B$ for B in the previous expansion.

Thus $(A-B)^3=A^3+3A^2(-B)+3A(-B)^2+(-B)^3$
$$=\mathbf{A^3-3A^2B+3AB^2-B^3}.$$

Example 11. Factorise $27x^3-8y^3$.

$$27x^3-8y^3=(3x)^3-(2y)^3=(3x-2y)[(3x)^2+(3x)(2y)+(2y)^2]$$
$$=\mathbf{(3x-2y)(9x^2+6xy+4y^2)}.$$

Example 12. Factorise $x^2-y^2+2yz-z^2$.

$$x^2-y^2+2yz-z^2=x^2-(y^2-2yz+z^2)=x^2-(y-z)^2$$
$$=[x+(y-z)][x-(y-z)]=\mathbf{(x+y-z)(x-y+z)}.$$

Example 13. Factorise $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

Arrange in powers of a,

the expression $=a^2(b-c)+b^2c-b^2a+c^2a-c^2b$
$$=a^2(b-c)-a(b^2-c^2)+b^2c-bc^2$$
$$=a^2(b-c)-a(b+c)(b-c)+bc(b-c)$$
$$=(b-c)[a^2-a(b+c)+bc]$$
$$=\mathbf{(b-c)(a-b)(a-c)}.$$

Expressions of this kind are usually arranged in cyclic order. Imagine the letters a, b, c arranged round a circle, and take the

differences obtained by writing the letters in the order in which they occur, round the circle; thus $a-b$, $b-c$, $c-a$. If an

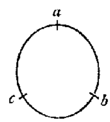


FIG. 10.

expression contains both $(b-c)$ and $(c-b)$ as factors, we retain $(b-c)$, but replace $(c-b)$ by $-(b-c)$; similarly we replace $(a-c)$ by $-(c-a)$.

$$\therefore\ a^2(b-c)+b^2(c-a)+c^2(a-b)=-(b-c)(c-a)(a-b).$$

This result may also be obtained by using the remainder theorem (compare Example 10 and Ex. VIII b, No. 35).

Example 14. Simplify

$$\frac{x^2}{(x-y)(x-z)}+\frac{y^2}{(y-z)(y-x)}+\frac{z^2}{(z-x)(z-y)}.$$

Here, we retain $(x-y)$, $(y-z)$, $(z-x)$ and replace $(y-x)$ by $-(x-y)$, $(z-y)$ by $-(y-z)$, $(x-z)$ by $-(z-x)$.

$$\begin{aligned}\therefore\ \text{the expression} &= -\frac{x^2}{(x-y)(z-x)}-\frac{y^2}{(y-z)(x-y)}-\frac{z^2}{(z-x)(y-z)}\\ &= -\frac{x^2(y-z)+y^2(z-x)+z^2(x-y)}{(x-y)(y-z)(z-x)}\\ &= -\frac{-(x-y)(y-z)(z-x)}{(x-y)(y-z)(z-x)}, \text{ by Example 13.}\\ &= -(-1)=1.\end{aligned}$$

EXERCISE VIII. c

Find the factors of the following:

1. b^3-1. 2. c^3+27. 3. $8-d^3$. 4. $8x^3+125y^3$.

5. x^4+x^2+1. 6. $3z^3-24$. 7. a^6+b^6. 8. x^4+4.

9. $a^2-b^2-c^2-2bc$. 10. $a^4-12a^2b^2+4b^4$.

11. $x^3-y^3-x^2y+xy^2$. 12. $x^2(1-z^2)+2xyz-y^2$.

13. x^3-3x^2+3x-1. 14. x^4-7x^2+1.

Simplify the following:

15. $\dfrac{x^3-y^3}{x-y}-\dfrac{x^3+y^3}{x+u}$. 16. $\dfrac{(a^3+b^3)(a-b)}{(a^3-b^3)(a+b)}$.

17. $\dfrac{x^3+y^3}{x^3-y^3}-\dfrac{x^2}{x^2+xy+y^2}$.

18. $\dfrac{a^4+a^2b^2+b^4}{(a^3-b^3)(a^3+b^3)}$.

19. $\dfrac{b^3-c^3}{b^2+c^2-2bc}-(b-c)$.

20. $\dfrac{(x^4-y^4)(x-y)}{(x^2-2xy+y^2)(x^2+y^2)}$.

21. Factorise (i) $x^3+(x-y)^3$; (ii) $(2a-b)^3-(a-2b)^3$.

22. Write down the squares of (i) $a-2b+c$; (ii) $x-3y-2z$.

23. Expand $(2x^2+3x+5)(2x^2-3x-5)$.

24. If $x^3-y^3=98$ and if $x-y=2$, *write down* the values of $x^2-2xy+y^2$, x^2+xy+y^2, xy, $x^2+2xy+y^2$, $x+y$. Hence find x and y.

25. If $x^3+y^3=370$ and if $x+y=10$, *write down* the values of $x^2+2xy+y^2$, x^2-xy+y^2, xy, $x^2-2xy+y^2$, $x-y$. Hence find x and y.

Simplify the following :

26. $\left(3-\dfrac{24}{x+3}\right)\left(2+\dfrac{16}{x-5}\right)$.

27. $\dfrac{x^2+xy+y^2}{x+y}-\dfrac{x^2-xy+y^2}{x-y}$.

28. $\dfrac{a^2+b^2-c^2-2ab}{a^2+b^2+c^2+2bc-2ca-2ab}$.

29. $\dfrac{x^2-2x-35}{x^2+2x-15}-\dfrac{x^2-6x+8}{x^2-2x-8}$.

30. $(x^2-xy+y^2)\left(\dfrac{1}{x}+\dfrac{1}{y}\right)\div\left(\dfrac{x}{y^2}+\dfrac{y}{x^2}\right)$.

31. $\left(x-\dfrac{10}{x+3}\right)\div\left(x-8+\dfrac{30}{x+3}\right)$.

32. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}$.

33. $\left(\dfrac{a+c}{b}-\dfrac{b+c}{a}\right)\div\left(1-\dfrac{a}{b}\right)$.

34. $\dfrac{x-y}{x^2+2xy+y^2}+\dfrac{x+y}{x^2-2xy+y^2}-\dfrac{2x}{x^2-y^2}$.

35. $(a+b-c)^2+(a-b+c)^2+(a-b-c)^2+(a+b+c)^2$.

36. $\dfrac{y+a}{(x-y)(z-y)}+\dfrac{z+a}{(x-z)(y-z)}$.

37. $\dfrac{x^3-2x^2-5x+6}{x^3+2x^2-x-2}$.

38. Solve $\dfrac{2}{x+1}+\dfrac{1}{x-2}=\dfrac{2}{x+3}+\dfrac{1}{x-6}$.

39. Solve $\dfrac{x-2}{x^2+2x}-\dfrac{x+2}{x^2+x}=\dfrac{x}{x^2+3x+2}$.

40. If $x=a^2+2ab-b^2$, $y=b^2+2ab-a^2$, $z=a^2+b^2$, prove that $x^2+y^2=2z^2$.

41. Prove that

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-yz-zx-xy)$$
$$=\tfrac{1}{2}(x+y+z)\{(y-z)^2+(z-x)^2+(x-y)^2\}.$$

42. Use No. 41 to factorise $a^3+b^3-c^3+3abc$.

43. If $x+y+z=0$, prove that $x^3+y^3+z^3=3xyz$.

Simplify the following :

44. $\dfrac{8b^3}{(a^2-b^2)(a^2+3b^2)}+\dfrac{2b}{a^2+3b^2}+\dfrac{1}{a+b}$.

45. $\left(\dfrac{1}{x}-\dfrac{1}{y}\right)\left(\dfrac{x}{x-y}+\dfrac{x}{x+y}\right)\div\left(\dfrac{y^2}{x+y}-\dfrac{y^2}{x-y}\right)$.

46. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}$.

47. $\dfrac{x-y}{(x+a)(a+y)}+\dfrac{y-z}{(y+a)(a+z)}+\dfrac{z-x}{(z+a)(a+x)}$.

48. $\dfrac{a^2}{(b-a)(c-a)}+\dfrac{b^2}{(c-b)(a-b)}+\dfrac{ac}{(a-c)(b-c)}$.

49. $\dfrac{a^2-x^2}{x^2+x(a-p)-ap}+\dfrac{b^2-x^2}{x^2+x(b-p)-bp}+\dfrac{x^2-c^2}{x^2+x(c-p)-cp}$.

50. Solve $x^4+x^2y^2+y^4=91$, $x^2+xy+y^2=13$.

[*Note. For further practice, see Appendix, Ex. F.P. 7, p. 160.*]

Undetermined Coefficients

If two expressions are *identically equal*, that is, are equal for all values of the letters they contain, they must, when simplified, agree term by term.

Example 15. Given that $4x^4-12x^3-11x^2+30x+25$ is a perfect square, find its square root.

The square root must be of the form, ax^2+bx+c; we have to find the values of the coefficients a, b, c, as yet undetermined.

$$4x^4-12x^3-11x^2+30x+25\equiv(ax^2+bx+c)^2$$
$$\equiv a^2x^4+2abx^3+\ldots+2bcx+c^2.$$

We must therefore choose a, b, c so that

$$a^2=4,\ 2ab=-12,\ 2bc=30.$$

(i) Take $a=2$, then $b=-3$ and $c=-5$.

(ii) Take $a=-2$, then $b=3$ and $c=5$.

$\therefore$ the square root is $2x^2-3x-5$ or $-2x^2+3x+5$,

that is, $\pm(2x^2-3x-5)$.

Check by showing that the coefficients of x^2 agree and that the constant terms agree, when $2x^2-3x-5$ is squared.

The work could be shortened by saying that *one* of the square roots of $4x^4 - 12x^3 \ldots$ must be of the form, $2x^2 + bx + c$.

Example 16. Given that $6x^2 + 11xy - 10y^2 - 4x - 29y - 10$ has linear factors in x and y, find them.

$$6x^2 + 11xy - 10y^2 \equiv (3x - 2y)(2x + 5y);$$

$$\therefore\ 6x^2 + 11xy - 10y^2 - 4x - 29y - 10 \equiv (3x - 2y + a)(2x + 5y + b),$$

where a, b are constants, whose values are as yet undetermined.

$$\therefore\ 6x^2 + 11xy - 10y^2 - 4x - 29y - 10$$
$$\equiv 6x^2 + 11xy - 10y^2 + x(3b + 2a) + y(-2b + 5a) + ab.$$

We must therefore choose a, b so that

$$3b + 2a = -4 \text{ and } -2b + 5a = -29.$$

Solving, $\quad 6b + 4a = -8$ and $-6b + 15a = -87$;

$$\therefore\ 19a = -95, \qquad \therefore\ a = -5;$$

$$\therefore\ 3b - 10 = -4; \quad \therefore\ 3b = 6; \quad \therefore\ b = 2.$$

$$\therefore\ 6x^2 + 11xy - 10y^2 - 4x - 29y - 10 \equiv (3x - 2y - 5)(2x + 5y + 2).$$

Check by showing that the constant terms agree.

Example 17. Express $\dfrac{x-8}{x^2-4}$ in the form $\dfrac{a}{x+2} + \dfrac{b}{x-2}$, where a, b are independent of x.

$$\frac{x-8}{x^2-4} \equiv \frac{x-8}{(x+2)(x-2)} \equiv \frac{a}{x+2} + \frac{b}{x-2}.$$

$$\therefore\ x - 8 \equiv a(x-2) + b(x+2) \equiv ax - 2a + bx + 2b$$
$$\equiv x(a+b) + (2b - 2a).$$

We must therefore choose a, b so that

$$a + b = 1 \text{ and } 2b - 2a = -8.$$

Solving, we have $a + b = 1$, $b - a = -4$.

$$\therefore\ 2b = -3; \quad \therefore\ b = -1\tfrac{1}{2},$$

and $\qquad 2a = 5; \quad \therefore\ a = 2\tfrac{1}{2};$

$$\therefore\ \frac{x-8}{x^2-4} \equiv \frac{2\frac{1}{2}}{x+2} - \frac{1\frac{1}{2}}{x-2}.$$

Instead of "equating coefficients," we can substitute any convenient values for x, since the expressions are identical.

In $(x-8) \equiv a(x-2) + b(x+2)$, put $x = 2$;

$$\therefore\ 2 - 8 = 0 + 4b; \qquad \therefore\ b = -1\tfrac{1}{2}.$$

Also put $x = -2$; $\therefore\ -2 - 8 = -4a + 0$; $\therefore\ a = 2\tfrac{1}{2}$, as before.

EXERCISE VIII. d

The expressions in Nos. 1-6 are perfect squares; find their square roots.

1. $x^4 - 6x^3 + 17x^2 - 24x + 16$.
2. $x^4 + 8x^3 + 14x^2 - 8x + 1$.
3. $9x^4 - 24x^3 + 28x^2 - 16x + 4$.
4. $9 - 30x + x^2 + 40x^3 + 16x^4$.
5. $4x^2 - 12x - 7 + \dfrac{24}{x} + \dfrac{16}{x^2}$.
6. $x^6 - 6x^5 + 7x^4 + 10x^3 - 11x^2 - 4x + 4$.

Find linear factors of the following:

7. $x^2 - 3xy + 2y^2 - 2x + 5y - 3$.
8. $x^2 - y^2 + 2x - 14y - 48$.
9. $6x^2 + 7xy - 20y^2 - 5x + 22y - 6$.
10. $3x^2 - 6y^2 - 2z^2 - 7yz - 5zx - 7xy$.

11. Find the values of a, b if $x^4 - 6x^3 + ax^2 + 30x + b$ is a perfect square.

12. Given that $8x^3 - 36x^2 + 54x - 27$ is a perfect cube, find its cube root.

13. Find the values of a, b if $27x^3 + ax^2 + bx - 125$ is a perfect cube.

14. For what value of c has $2x^2 + 3xy - 2y^2 + 5x - 5y + c$ linear factors? What are then the factors?

15. Express $\dfrac{x-8}{(x+1)(x-2)}$ in the form $\dfrac{a}{x+1} + \dfrac{b}{x-2}$.

16. For what values of b, c is $x^3 + 12x^2 + bx + c$ a perfect cube?

17. Express $n^2 - 2n$ in the form, $an(n+1) + b(n+1) + c$.

18. Express $\dfrac{5x-4}{x(x+1)(x-2)}$ in the form, $\dfrac{a}{x} + \dfrac{b}{x+1} + \dfrac{c}{x-2}$.

19. Express $4n^2 - 1$ in the form, $a(n+1)^2 + b(n+1) + c$.

20. For what values of b, c is $x^2 - x + 1$ a factor of
$$x^3 + 3x^2 + bx + c\ ?$$

21. For what values of a, b is $x - y + 2$ a factor of
$$x^2 + 2xy - 3y^2 + ax + by - 2.$$
What is then the other factor?

22. Express $\dfrac{5x^2 - 3x + 4}{(x+1)^2(x-2)}$ in the form, $\dfrac{a}{x+1} + \dfrac{b}{(x+1)^2} + \dfrac{c}{x-2}$.

23. Express n^3 in the form, $a + bn + cn(n-1) + dn(n-1)(n-2)$.

24. Show that $(x+y+z)^3 - (x^3+y^3+z^3)$ can be put in the form, $c(x+y)(y+z)(z+x)$; and then find the value of c.

The Binomial Theorem

Successive Powers of $1+x$. To work out quickly the expansions $(1+x)^2$, $(1+x)^3$, $(1+x)^4$, ... , use detached coefficients.

$$\begin{array}{l}
(1+x)^2\\
1+1\\
\underline{1+1}\\
1+1\\
\underline{\quad 1+1}\\
\underline{1+2x+x^2}
\end{array}
\qquad
\begin{array}{l}
(1+x)^3\\
1+2+1\\
\underline{1+1\qquad}\\
1+2+1\\
\underline{\quad 1+2+1}\\
\underline{1+3x+3x^2+x^3}
\end{array}
\qquad
\begin{array}{l}
(1+x)^4\\
1+3+3+1\\
\underline{1+1\qquad\qquad}\\
1+3+3+1\\
\underline{\quad 1+3+3+1}\\
\underline{1+4x+6x^2+4x^3+x^4}
\end{array}$$

From this work, it is clear that the coefficients of any expansion are formed by adding together two consecutive coefficients in the preceding expansion.

Thus $(1+x)^5 = 1+(1+4)x+(4+6)x^2+(6+4)x^3+(4+1)x^4+x^5$;

$\therefore\ (1+x)^5 = 1+5x+10x^2+10x^3+5x^4+x^5.$

By the same method, we can *write down* the expansion of $(1+x)^6$.

$$(1+x)^6 = 1+6x+15x^2+20x^3+15x^4+6x^5+x^6.$$

The reader should now *write down in succession* the expansions of $(1+x)^7$, $(1+x)^8$, $(1+x)^9$, etc.

Pascal's Triangle. The coefficients in these successive expansions may be written as follows:

$(1+x)^0$	1						
$(1+x)^1$	1	1					
$(1+x)^2$	1	2	1				
$(1+x)^3$	1	3	3	1			
$(1+x)^4$	1	4	6	4	1		
$(1+x)^5$	1	5	10	10	5	1	
$(1+x)^6$	1	6	15	20	15	6	1

....

In this arrangement, called Pascal's triangle, each number is the sum of two numbers in the previous row, the number just above it and the next number to the left of that; thus in the last row $15=5+10$, $20=10+10$. It follows that each number is the sum of *all* the numbers in the *next column* to the left *above* it; thus we have in succession

$$15=5+10=5+4+6=5+4+3+3=5+4+3+2+1$$

and $\quad 20=10+10=10+6+4=10+6+3+1$; etc.

There are n rows above the row containing the coefficients for $(1+x)^n$; therefore this row starts

$$1, n, \dots .$$

Therefore the third number in this row is the sum of

$$1, 2, 3, \dots, n-2, n-1,$$

which equals $\frac{n(n-1)}{2}$ and is written $\frac{n(n-1)}{1.2}$.

Therefore the fourth number in this row is the sum of

$$\frac{2.1}{2}, \frac{3.2}{2}, \frac{4.3}{2}, \dots, \frac{(n-2)(n-3)}{2}, \frac{(n-1)(n-2)}{2}.$$

These numbers may be written

$$\frac{3.2.1}{1.2.3}, \frac{4.3.2}{1.2.3}-\frac{3.2.1}{1.2.3}, \frac{5.4.3}{1.2.3}-\frac{4.3.2}{1.2.3}, \dots$$

the last two numbers being

$$\frac{(n-1)(n-2)(n-3)}{1.2.3}-\frac{(n-2)(n-3)(n-4)}{1.2.3} \text{ and}$$
$$\frac{n(n-1)(n-2)}{1.2.3}-\frac{(n-1)(n-2)(n-3)}{1.2.3};$$

$\therefore$ the sum of all these numbers is $\frac{n(n-1)(n-2)}{1.2.3}$;

$\therefore$ the row of coefficients for $(1+x)^n$ runs

$$1, n, \frac{n(n-1)}{1.2}, \frac{n(n-1)(n-2)}{1.2.3}, \dots .$$

By the same method we can show that the next number in the row is $\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$, and so on.

The general formula is as follows:

If n is any positive integer,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \dots$$
$$+\frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r}x^r + \dots + x^n$$

and similarly, if n is any positive integer,

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{1.2}x^{n-2}y^2$$
$$+\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}y^3 + \dots + y^n.$$

This formula is called the *Binomial Theorem* for a positive integral index.

The reader should note that the rth term in the expansion of $(x+y)^n$ involves y^{r-1}, therefore the degree of x is $n-(r-1)$; also the denominator is $1 . 2 . 3 . 4 ... (r-1)$, and there are the same number of factors, namely $r-1$, in the numerator.

Example 18. Write down the expansion of $(3a-\frac{1}{2}b)^4$.

$$(3a-\tfrac{1}{2}b)^4=(3a)^4+4(3a)^3(-\tfrac{1}{2}b)+\frac{4 . 3}{1 . 2}(3a)^2(-\tfrac{1}{2}b)^2$$
$$+\frac{4 . 3 . 2}{1 . 2 . 3}(3a)(-\tfrac{1}{2}b)^3+(\tfrac{1}{2}b)^4$$
$$=81a^4-54a^3b+\tfrac{27}{2}a^2b^2-\tfrac{3}{2}ab^3+\tfrac{1}{16}b^4.$$

Example 19. Write down the 5th term in the expansion of $(2a-3b)^{10}$.

The 5th term involves $(-3b)^4$; $\therefore$ it also contains $(2a)^6$; and its denominator is $1 . 2 . 3 . 4$; $\therefore$ its numerator is $10 . 9 . 8 . 7$;

$$\therefore \text{ the 5th term}=\frac{10 . 9 . 8 . 7}{1 . 2 . 3 . 4}(2a)^6(-3b)^4.$$

Example 20. Find the term independent of x in the expansion of $\left(3x-\frac{5}{x^3}\right)^8$.

$$\left(3x-\frac{5}{x^3}\right)^8=\left(\frac{3x^4-5}{x^3}\right)^8=\frac{(3x^4-5)^8}{x^{24}};$$

$\therefore$ we want the term involving x^{24} in $(3x^4-5)^8$.

This term is $\frac{8 . 7}{1 . 2}(3x^4)^6(-5)^2$

$$=700\times 3^6x^{24};$$

$\therefore$ the term independent of x is 700×3^6.

Factorial r. The product of the first r positive integers,

$$1 . 2 . 3 . 4 ... (r-1)r$$

is denoted by the symbol $r!$ and is called "factorial r."

In mss., as opposed to print, it is often written $\underline{|r}$.

The coefficient of x^r in $(1+x)^n$ is

$$\frac{n(n-1)(n-2)...(n-r+1)}{1 . 2 . 3 ... (r-1)r};$$

if we multiply numerator and denominator by

$$(n-r)(n-r-1)(n-r-2)...3 . 2 . 1,$$

that is, by $(n-r)!$, the fraction becomes $\frac{n!}{r!(n-r)!}$; for some purposes, this is a more convenient form.

Example 21. Write down the coefficient of x^7 in $(3-5x)^{12}$.

The coefficient is $\frac{12!}{7!5!}3^5 \cdot (-5)^7$.

This reduces to $-2^3 \cdot 3^7 \cdot 5^7 \cdot 11$.

EXERCISE VIII.

[*Coefficients which are large should be given in factors, not multiplied out.*]

Write down the expansions of Nos. 1-8.

1. $(1-a)^4$. 2. $(b+1)^5$. 3. $(1-2c)^3$. 4. $(d+2)^4$.

5. $(x-y)^6$. 6. $\left(x+\frac{1}{x}\right)^7$. 7. $(2x-3y)^4$. 8. $(a^2-3b)^5$.

9. In the expansion of $(1+x)^7$, (i) What is the number of terms? (ii) Which term involves x^3? (iii) What is the power of x in the 6th term? (iv) What is the coefficient of x^4?

10. In the expansion of $(4x-5y)^{12}$,
   (i) What is the number of terms?
   (ii) What powers of x and y occur in the 3rd term? in the 11th term?
   (iii) If one term is ax^py^q, what do you know about p and q?
   (iv) What is the last term? the 7th term?

11. What is the coefficient of x^3 in
   (i) $(1+2x)^5$; (ii) $(1-x)^7$; (iii) $(3-x)^{10}$; (iv) $(x+2)^8$?

12. What is the coefficient of x^4 in (i) $(2-\frac{1}{2}x)^8$; (ii) $(3x-1)^6$?

13. What is the coefficient of x^7 in (i) $(1-3x)^{10}$; (ii) $(4x-5)^{15}$?

14. What is the coefficient of x^8 in (i) $(2x-x^2)^6$; (ii) $(3-x^2)^7$?

15. What is the 4th term in the expansion of
   (i) $(1-x)^{12}$; (ii) $(2-x)^{10}$; (iii) $(1+\frac{1}{2}x)^n$?

16. What is the 7th term in the expansion of
   (i) $(x-y)^9$; (ii) $(2-3x)^{10}$; (iii) $(5a-3b)^{12}$?

17. What is (i) the 5th term (ii) the rth term in $(1+2x)^n$?

18. What is (i) the 4th term from the beginning, (ii) the 4th term from the end in the expansion of $(x+y)^{10}$?

19. What is the term independent of x in

$$\text{(i) } \left(x+\frac{2}{x}\right)^4; \quad \text{(ii) } \left(3x-\frac{5}{x}\right)^6 ?$$

20. How many terms are there in the expansion of $(2a-3b)^6$? What is the middle term?

21. What is the greatest coefficient in the expansion of

(i) $(1+x)^8$; (ii) $(1+x)^9$?

22. Using factorials write down the coefficients of the named terms in the following:

(i) x^8 in $(2-x)^{11}$; (ii) x^9 in $(4-3x)^{13}$;

(iii) x^n in $(1+x)^{2n}$; (iv) x^{n-1} and x^n in $(1+x)^{2n-1}$;

(v) x^{14} in $(3x+2x^2)^9$; (vi) x^{10} in $(1-2x^2)^8$;

(vii) x^4 in $\left(x+\frac{1}{x}\right)^{12}$; (viii) $\frac{1}{x^3}$ in $\left(x-\frac{1}{x}\right)^5$.

23. What is (i) the 4th term, (ii) the rth term in $(1-x^2)^n$?

24. Simplify $(a-1)^3+3(a-1)^2+3(a-1)+1$.

25. Simplify $1-4(b+1)+6(b+1)^2-4(b+1)^3+(b+1)^4$.

26. What is the coefficient of x^4 in $(1+x+x^2)^{10}$?

27. What is the coefficient of x^3 in $(1+x+x^2)^n$?

28. What is the nth term in the expansion of $(1+x^2)^{3n}$?

29. What are the coefficients of x^4 and x^5 in $(1+x)^7$? Deduce the coefficient of x^5 in $(1-x)(1+x)^7$.

30. What are the coefficients of x^2 and x^3 in $(1-x)^n$? Deduce the coefficient of x^3 in $(1+x)(1-x)^n$.

31. Evaluate $(\sqrt{2}+1)^6+(\sqrt{2}-1)^6$.

32. Find the term independent of x in the expansion of

$$\left(x+\frac{1}{x}\right)^{2n}.$$

33. Express compactly by using factorials:

(i) $12.11.10.9.8.7$; (ii) $20.18.16.14.12.10$;

(iii) $2.4.6.8\ldots(2n-2).2n$;

(iv) $1.3.5.7\ldots(2n-1)$.

34. Simplify (i) $n!\div(n-3)!$; (ii) $n!-(n-1)!$.

35. If $20!$ is expressed in prime factors, to what power will (i) 2, (ii) 3 occur?

36. Simplify (i) $\dfrac{n!}{r!\,(n-r)!}+\dfrac{n!}{(r-1)!\,(n-r+1)!}$;

(ii) $\dfrac{n!}{r!\,(n-r)!}\div\dfrac{n!}{(r-1)!\,(n-r+1)!}$.

37. Prove that the coefficient of $x^n\,y^n\,z^n$ in the expansion of $(x+y+z)^{3n}$ is $(3n)!/(n!)^3$.

38. Prove that the coefficient of x^3 in the expansion of $(3+x)\,(1+x)^n$ is $\frac{1}{2}n(n-1)^2$.

CHAPTER IX

LITERAL EQUATIONS AND QUADRATIC FUNCTIONS

Literal Equations

Use the same methods as are used for numerical equations

Simple Equations

Collect on the left-hand side terms which contain the unknown, and remove all other terms to the right-hand side.

Example 1. Solve, for x, $ax+\frac{b}{a}=bx+\frac{a}{b}$ where $a\neq b$.

Subtract $\frac{b}{a}$ and bx from each side,

$$\therefore\ ax-bx=\frac{a}{b}-\frac{b}{a};$$

$$\therefore\ x(a-b)=\frac{a^2-b^2}{ab}=\frac{(a+b)(a-b)}{ab}.$$

Divide each side by $(a-b)$, since $a-b\neq 0$,

$$\therefore\ x=\frac{(a+b)(a-b)}{ab(a-b)}=\frac{\mathbf{a+b}}{\mathbf{ab}}.$$

EXERCISE IX. a

Solve, for x, the following equations:

1. $b+x=c$. 2. $bx=c$. 3. $\frac{1}{x}=a$.

4. $\frac{a}{x}=\frac{b}{c}$. 5. $a-x=\frac{1}{b}$. 6. $\frac{1}{x}=b+c$.

7. $x=c-x$. 8. $2x-a=a-b$. 9. $\frac{x}{a}=\frac{a}{b}$.

10. $bx=b+b^2$. 11. $\frac{x}{a}=1-a$. 12. $p(x+q)=3pq$.

13. $lx+bx=A$. 14. $\frac{1}{x}=\frac{1}{a}+\frac{1}{b}$. 15. $\frac{x}{2a}+\frac{x}{3a}=b$.

16 $\frac{c}{x}+\frac{d}{x}=\frac{1}{2}$. 17 $nx+x=k$. 18. $x=mx+c$.

19. $\frac{x+p}{x-q}=2$. 20 $x-\frac{x}{c}=d$. 21. $\frac{1}{x+r}=s$.

22. $px+q=p+qx$. 23. $x(a+b)=c(ab-x)$.

24. $ax+bx=a^2-b^2$. 25. $p(x-p)=q(x-q)$.

26. $r(x+s)=s(x+r)$. 27. $c(x-d)=d(x+c)$.

28. $\frac{x+m}{n}=\frac{x+n}{m}$. 29. $(x-a)^2=x^2+b^2$.

30. $\frac{1}{x-a}+\frac{1}{x-b}=\frac{2}{x}$. 31. $(x-p)(x+q)=(x+p)(x-q)$.

32. $\frac{c}{x-d}+\frac{d}{x-c}=\frac{c+d}{x}$. 33. $\frac{c-b}{x-a}=\frac{c-a}{x-b}$.

[*Note. For further practice, see Appendix, Ex. F.P.* 8, *p.* 161.]

Simultaneous Simple Equations

Use the method of addition or subtraction.

Example 2. Find x and y from the simultaneous equations,

$$ax+by+c=0 \quad \text{.................(i)}$$
$$px+qy+r=0 \quad \text{.................(ii)}$$

Multiply each side of (i) by q and each side of (ii) by b;

$$\therefore\ aqx+bqy+cq=0, \quad \text{.................(iii)}$$
$$bpx+bqy+br=0. \quad \text{.................(iv)}$$

From (iii) and (iv), by subtraction,

$$aqx-bpx+cq-br=0;$$

$$\therefore\ aqx-bpx=br-cq; \qquad \therefore\ x(aq-bp)=br-cq;$$

$$\therefore\ x=\frac{br-cq}{aq-bp}.$$

The value of y can be found by substituting this value of x in (i); but it is *easier* to find y by starting again with (i) and (ii) and eliminating x. The reader should do this; he will find that

$$y=\frac{cp-ar}{aq-bp};$$

$$\therefore \text{ the solution is } \mathbf{x}=\frac{\mathbf{br-cq}}{\mathbf{aq-bp}},\ \mathbf{y}=\frac{\mathbf{cp-ar}}{\mathbf{aq-bp}}.$$

This is *a formula* for solving any pair of simultaneous equations of the first degree, in two unknowns. The answer can be written in the form :

$$\frac{x}{br-cq}=\frac{y}{cp-ar}=\frac{1}{aq-bp}.$$

At first, simultaneous simple equations should be solved from first principles, **not** *by quoting a formula* ; but it is useful for the reader to see a more symmetrical method of writing this result.

Many formulae can be expressed in a symmetrical form, if the following notation is used.

$\begin{vmatrix} A & B \\ P & Q \end{vmatrix}$ stands for the "cross-product," $AQ-BP$.

It is called a *determinant.*

Using this notation, we see that the roots of the simultaneous equations,

$$\left.\begin{aligned} ax+by+c&=0 \\ px+qy+r&=0 \end{aligned}\right\}$$

are given by

$$\frac{x}{\begin{vmatrix} b & c \\ q & r \end{vmatrix}}=\frac{y}{\begin{vmatrix} c & a \\ r & p \end{vmatrix}}=\frac{1}{\begin{vmatrix} a & b \\ p & q \end{vmatrix}}$$

The next example shows how time can often be saved by simplifying **before starting** to eliminate one unknown.

Example 3. Find x and y from the simultaneous equations,

$$(a+b)x+(a-b)y=a^2-b^2 \quad \text{.....................(i)}$$
$$(a-b)x-(a+b)y=-2ab \quad \text{.....................(ii)}$$

By addition, from (i) and (ii),

$$2ax-2by=a^2-b^2-2ab \quad \text{.....................(iii)}$$

By subtraction, from (i) and (ii),

$$2bx+2ay=a^2-b^2+2ab \quad \text{.....................(iv)}$$

Multiply each side of (iii) by a and each side of (iv) by b.

$$\therefore\ 2a^2x-2aby=a^3-ab^2-2a^2b \quad \text{..................(v)}$$

and

$$2b^2x+2aby=a^2b-b^3+2ab^2 \quad \text{..................(vi)}$$

By addition, from (v) and (vi),

$$2a^2x+2b^2x=a^3-a^2b+ab^2-b^3;$$

$$\therefore\ 2x(a^2+b^2)=a^2(a-b)+b^2(a-b)=(a-b)(a^2+b^2);$$

$$\therefore\ x=\frac{(a-b)(a^2+b^2)}{2(a^2+b^2)}=\frac{a-b}{2}.$$

Substitute for x in (iv),

$$\therefore \frac{2b(a-b)}{2}+2ay=a^2-b^2+2ab;$$

$$\therefore ab-b^2+2ay=a^2-b^2+2ab; \quad \therefore 2ay=a^2+ab=a(a+b);$$

$$\therefore y=\frac{a(a+b)}{2a}=\frac{a+b}{2};$$

$\therefore$ the solution is $\mathbf{x}=\frac{1}{2}(\mathbf{a}-\mathbf{b})$, $\mathbf{y}=\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

EXERCISE IX. b

Find x and y from the following simultaneous equations :

1. $x+y=7c$, $x-y=3c$.
2. $2x-y=a$, $x+3y=b$.
3. $ax+2by=4c$, $2ax-by=3c$.
4. $x+y=p+q$, $px+qy=2pq$.
5. $x+cy=d$, $x+dy=c$.
6. $ax+y=ab$, $bx-ay=b^2$.
7. $\frac{x+y}{a}=\frac{x-y}{b}=k$.
8. $\frac{a}{x+b}=\frac{b}{y+a}=\frac{c}{d}$.
9. $ax-by=a^2+b^2$, $x+y=2a$.
10. $cx-dy=0$, $x+y=k$.
11. $\frac{a}{x}+\frac{b}{y}=c$, $\frac{a}{x}-\frac{b}{y}=d$.
12. $b(x-b)=ay$, $a(x-a)=b(x-y)$.
13. $cx+dy=2cd$, $dx-cy=d^2-c^2$.
14. $y=mx+c$, $x=my+c$.
15. $ax+by=bx-ay=a^2+b^2$.
16. $x-p=y-q=\frac{px-qy}{p+q}$.
17. $qx=py$, $px-qy=p+q$.
18. $\frac{x}{a}+\frac{y}{b}=2$, $ax+by=a^2+b^2$.
19. $\frac{x-a}{b}+\frac{y}{a}=0$, $\frac{x-b}{a}-\frac{y}{b}=0$.
20. $x(c+d)+y(1-c)=d(c+d)$, $x(c-d)+y(1+c)=d(c-d)$.
21. $(p-q)x+(p+q)y=(p+q)^2$, $qx-py=q^2-pq$.
22. $r(x+y)+s(x-y)=2rs$, $s(x+y)-r(x-y)=s^2-r^2$.
23. $bx-(a+b)y=ab-b^2$, $(a-b)x-2ay=2b^2$.
24. $x(c+1)+y(1-d)=x(1-c)+y(1+d)=k$.

Write down by means of the formula the solution of the following :

25. $3x+5y+2=0$, $2x+3y+4=0$.
26. $4x+3y-3=0$, $3x+4y-2=0$.
27. $5x-2y-25=0$, $4x+3y+3=0$.

28. $x - 3y = 1,$
$2x + y = 16.$

29. $6x + 5y = -7,$
$4x - 3y = 8.$

30. $10x - 6y = 3,$
$14x - 9y = 4.$

[*For further practice in literal equations and literal relations, see Appendix, Ex. F.P.* 9, 10, pp. 163-5 *and Ex. T.* 8, *p.* 177.]

Quadratic Equations

Any quadratic equation can be reduced to the form

$$ax^2 + bx + c = 0.$$

And it has been proved (Pt. II, Ch. XIV) that its roots are

$$\frac{-b + \sqrt{(b^2 - 4ac)}}{2a}, \quad \frac{-b - \sqrt{(b^2 - 4ac)}}{2a},$$

unless $b^2 < 4ac$.

The connection between the roots and the coefficients was indicated in Pt. II, Ch. XII. This work should now be revised and extended.

Denote by α, β the two roots of $ax^2 + bx + c = 0$. Then the equation is

$$(x - \alpha)(x - \beta) = 0 \quad \text{or} \quad x^2 - \alpha x - \beta x + \alpha\beta = 0.$$

$\therefore$ $x^2 - x(\alpha + \beta) + \alpha\beta = 0$ and $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ are equivalent equations. This requires that

$$\boldsymbol{\alpha + \beta = -\frac{b}{a}} \quad \text{and} \quad \boldsymbol{\alpha\beta = \frac{c}{a}}.$$

These two facts should be committed to memory; they may be stated as follows:

If all terms of a quadratic equation are brought to one side, and, *if the coefficient of x^2 is* $+1$,

(i) **the sum of the roots = the coefficient of x with the sign changed;**

(ii) **the product of the roots = the constant term.**

Nature of Roots of a Quadratic. The forms of the two roots of $ax^2 + bx + c = 0$ lead to the following conclusions.

(i) If $b^2 - 4ac$ is a perfect square and if a, b, c are rational, the roots are rational and unequal.

(ii) If $b^2 - 4ac > 0$, but is not a perfect square, the roots are irrational and unequal.

(iii) If $b^2 - 4ac = 0$, the roots are equal.

(iv) If $b^2 - 4ac < 0$, there are no roots.

Note. It is (unfortunately) customary to say that the square root of a negative number is an "imaginary number," and, in contrast, that the square root of a positive number is a "real number." And so the equation $ax^2+bx+c=0$ is said to have "real roots" if $b^2-4ac>0$, and to have "imaginary roots" if $b^2-4ac<0$.

Example 4. Find the sum of the squares of the roots of the equation $5x^2-x-3=0$.

Let α, β be the roots of $5x^2-x-3=0$ or $x^2-\frac{1}{5}x-\frac{3}{5}=0$.

$$\therefore\ \alpha+\beta=+\tfrac{1}{5} \text{ and } \alpha\beta=-\tfrac{3}{5}.$$

$$\therefore\ \alpha^2+2\alpha\beta+\beta^2=(\alpha+\beta)^2=\tfrac{1}{25}\ ;\ \text{ but } 2\alpha\beta=-\tfrac{6}{5}\ ;$$

$$\therefore\ \alpha^2+\beta^2=\frac{1}{25}+\frac{6}{5}=\frac{31}{25}.$$

Example 5. If one root of $x^2+px+36=0$ is four times the other, find the value of p.

Let the roots be α, 4α ; $\therefore$ $5\alpha=-p$ and $4\alpha^2=36$.

$$\therefore\ \alpha^2=9\ ;\quad \therefore\ \alpha=\pm 3\ ;\quad \therefore\ p=\mp 15.$$

Example 6. If α, β are the roots of $ax^2+bx+c=0$,

(i) express $\alpha^3+\beta^3$ in terms of a, b, c ;

(ii) form the equation whose roots are $\alpha-2\beta$, $\beta-2\alpha$.

From the data, $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$.

(i) $\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)=(\alpha+\beta)\{(\alpha+\beta)^2-3\alpha\beta\}$

$$=\left(-\frac{b}{a}\right)\left\{\frac{b^2}{a^2}-\frac{3c}{a}\right\}=-\frac{\mathbf{b(b^2-3ac)}}{\mathbf{a^3}}.$$

(ii) Let the required equation be $x^2+qx+r=0$;

then $\alpha-2\beta+\beta-2\alpha=-q$ and $(\alpha-2\beta)(\beta-2\alpha)=r$.

$$\therefore\ q=\alpha+\beta=-\frac{b}{a},$$

$$\text{and } r=-2\alpha^2+5\alpha\beta-2\beta^2=-2(\alpha+\beta)^2+9\alpha\beta$$

$$=-\frac{2b^2}{a^2}+\frac{9c}{a}=\frac{9ac-2b^2}{a^2}\ ;$$

$$\therefore\ \text{the equation is } x^2-\frac{b}{a}x+\frac{9ac-2b^2}{a^2}=0,$$

or
$$\mathbf{a^2x^2-abx+9ac-2b^2=0}.$$

Note. In questions involving the roots of a quadratic, avoid when possible treating each root separately. Use the values of $\alpha+\beta$ and $\alpha\beta$, rather than the separate values of α and of β.

EXERCISE IX. c

1. What is the nature of the roots of the following equations The actual roots are not required.

(i) $x^2-3x-7=0$; (ii) $x^2-3x+7=0$;
(iii) $x^2-3x-10=0$; (iv) $x^2-3x+2\frac{1}{4}=0$;
(v) $5x^2+7x+3=0$; (vi) $24x^2-35x-150=0$;
(vii) $9x^2-24x+16=0$; (viii) $3x^2+11x+9=0$.

2. Form the equations whose roots are

(i) $3,\ -7$; (ii) ± 4; (iii) $\pm\sqrt{3}$;
(iv) $3+\sqrt{2},\ 3-\sqrt{2}$; (v) $2p,\ 3p$;
(vi) $-5p,\ p$; (vii) $1,\ \frac{c}{a}$; (viii) $\frac{p}{q},\ -\frac{r}{s}$.

3. Find the greatest value of c for which (i) $x^2-6x+c=0$; (ii) $x^2+5x+c=0$ has roots.

4. If $x=3$ is a root of each of the following equations, *write down* the other root, and then find the value of p.

(i) $x^2-5x+p=0$; (ii) $x^2+4x+p=0$;
(iii) $x^2+px+12=0$; (iv) $5x^2+px-6=0$.

5. One root of $7x^2+6x+c=0$ is double the other; find them; also find c.

6. Find c if $3x^2-8x+c=0$ has equal roots.

7. One root of $4x^2+bx+75=0$ is three times the other; find them; also find b.

8. If $\alpha,\ \beta$ are the roots of $x^2-4x-7=0$, find the values of

(i) $\alpha^2+\beta^2$; (ii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$; (iii) $\frac{1}{\alpha}+\frac{1}{\beta}$.

9. If $\alpha,\ \beta$ are the roots of $3x^2+2x-4=0$, find the values of

(i) $\alpha^2+\alpha\beta+\beta^2$; (ii) $\alpha^3+\beta^3$.

10. If $\alpha,\ \beta$ are the roots of $2x^2+6x+3=0$, form the equation whose roots are

(i) $\frac{1}{\alpha},\ \frac{1}{\beta}$; (ii) $\alpha^2,\ \beta^2$.

11. If $\alpha,\ \beta$ are the roots of $x^2+qx+r=0$, find the condition that

(i) $\alpha=\frac{1}{\beta}$; (ii) $\alpha=2\beta$; (iii) $\frac{1}{\alpha}+\frac{1}{\beta}=3$.

12. If $\alpha,\ \beta$ are the roots of $ax^2+bx+c=0$, form the equation whose roots are

(i) $\frac{1}{\alpha},\ \frac{1}{\beta}$; (ii) $\frac{\alpha}{\beta},\ \frac{\beta}{\alpha}$; (iii) $\alpha+\frac{1}{\beta},\ \beta+\frac{1}{\alpha}$.

13. What is the equation whose roots are 3 times the roots of $6x^2+2x-5=0$?

14. What is the equation whose roots are each 1 less than the roots of $2x^2-x=5$?

15. Find c if $x^2+(c-1)x=2c+1$ has equal roots.

16. What can you say about b if $x+\frac{1}{x}=b$ has roots ?

17. What can you say about k if $4+2x-x^2=k$ has roots ? Hence find the greatest value of $4+2x-x^2$.

18. What can you say about k if $3x^2-10x+7=k$ has roots ? Hence find the least value of $3x^2-10x+7$.

19. If α, β are the roots of $ax^2+bx+c=0$, find in terms of a, b, c,

$$\text{(i) } \frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}; \quad \text{(ii) } (\alpha-1)(\beta-1).$$

20. Prove that $ax^2-(a+c)x+c=0$ has rational roots.

Solve for x the following equations :

21. $x^2-3cx=10c^2$.

22. $(x-b)^2=4c^2$.

23. $x^2-4x=4(b^2-1)$.

24. $x^2+2px=q^2$.

25. $a(x^2+1)=xa(^2+1)$.

26. $x-\frac{1}{x}=c$.

27. Find the equation whose roots are the squares of the roots of $2x^2-4x=3$.

28. Find the condition that the roots of $x^2+qx+r=0$ differ by $3q$.

29. Prove that $(a+4b)x^2-2(a+b)x+a-2b=0$ has rational roots, if a and b are rational.

30. Prove that $(a^2+b^2)x^2-2(a+b)x+2=0$ has no roots, if a and b are unequal.

31. What is the condition that $x^2+xy-2y^2$ and $ax^2+2hxy+by^2$ have a common factor of the form $x+cy$?

32. What is the condition that the equations, $x^2+bx+c=0$ and $x^2-cx-b=0$, have a common root ?

33. If α and β are the roots of $px^2+qx+r=0$, simplify

$$(q\alpha+r)(q\beta+r).$$

34. Find c if $4(x-2)^2=c(5-8x)$ has equal roots.

35. Find c if the sum of the roots of $x^2-(c+6)x+2(2c-1)=0$ is half their product.

36. (i) Write down the equation whose roots are α, β, γ and simplify it out, collecting like terms in x.

(ii) If α, β, γ are the roots of $ax^3+bx^2+cx+d=0$, express $\alpha+\beta+\gamma$, $\beta\gamma+\gamma\alpha+\alpha\beta$, $\alpha\beta\gamma$ in terms of a, b, c, d.

37. (i) The roots of $x^3+px^2+qx+r=0$ are 3, 1, -2; what are the values of p, q, r?

(ii) Two of the roots of $x^3+qx+r=0$ are 4, 3; what is the other root? What are the values of q, r?

[*For further practice, see Appendix, Ex. F.P.* 10, *p.* 165.]

Quadratic Functions

We can transform the quadratic function ax^2+bx+c as follows:

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

$$=a\left\{x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right\}$$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right\}.$$

(i) If $b^2-4ac>0$, by using the method of "difference of two squares," we obtain two real unequal factors of ax^2+bx+c, *for all values of* x.

Thus, numbers α and β can be found such that

$$ax^2+bx+c=a(x-\alpha)(x-\beta)$$

for *all values* of x.

Therefore the function ax^2+bx+c is zero for two values of x, namely $x=\alpha$ and $x=\beta$, and is positive for some values of x, and negative for other values of x.

(ii) If $b^2-4ac=0$, ax^2+bx+c is a perfect square, viz. $a\left(x+\frac{b}{2a}\right)^2$ *for all values of* x.

In this case, ax^2+bx+c is zero for one value of x only, namely $x=-\frac{b}{2a}$, and for other values of x has the same sign as a.

(iii) If $b^2-4ac<0$,

$$ax^2+bx+c=a\left\{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right\},$$

and is therefore the "sum of two squares," and numbers p and q can be found such that

$$ax^2 + bx + c = a\{(x+p)^2 + q^2\}$$

for *all values* of x.

In this case $ax^2 + bx + c$ is never zero, and for all values of x has the same sign as a.

Rough graphs should now be drawn to illustrate these three cases (cf. Ex. VII. a. Nos. 8-12).

Example 7. For what range of values of x is the function $5 + 7x - 6x^2$ negative? Find also its greatest value.

(i) $5 + 7x - 6x^2 = (1 + 2x)(5 - 3x) = 2(\frac{1}{2} + x)(-3)(x - \frac{5}{3})$

$$= -6(x - \tfrac{5}{3})(x + \tfrac{1}{2});$$

$\therefore$ $5 + 7x - 6x^2$ is negative if $(x - \frac{5}{3})(x + \frac{1}{2})$ is positive, that is, if $x - \frac{5}{3}$ and $x + \frac{1}{2}$ are either *both* positive or *both* negative.

Both factors are positive if $x > \frac{5}{3}$.

Both factors are negative if $x < -\frac{1}{2}$.

$$\therefore\ 5 + 7x - 6x^2 \text{ is negative if } x > \tfrac{5}{3} \text{ or if } x < -\tfrac{1}{2}.$$

(ii) $5 + 7x - 6x^2 = 5 - 6\left(x^2 - \dfrac{7x}{6}\right)$

$$= 5 - 6(x - \tfrac{7}{12})^2 + 6 \times \tfrac{49}{144}$$

$$= 7\tfrac{1}{24} - 6(x - \tfrac{7}{12})^2.$$

Now the least value of $(x - \frac{7}{12})^2$ is zero;

$\therefore$ the greatest value of $5 + 7x - 6x^2$ is $7\frac{1}{24}$, and the function takes this value when $x = \frac{7}{12}$.

EXERCISE IX. d

1. For what ranges of values of x is (i) $(2x - 3)(x + 4)$ positive; (ii) $(3x - 5)(2x + 1)$ negative?

2. What is the least value of $3\{(2x - 1)^2 + 4\}$?

3. What is the greatest value of $2\frac{1}{4} - 3(x - 1)^2$?

4. Which of the following functions have greatest values:

(i) $(1 - x)^2$; (ii) $1 - x^2$;

(iii) $(5 - x)(4 - x)$; (iv) $(x - 5)(2 - x)$?

5. For what range of values of x is $3 - 2x - 8x^2$ positive?

6. For what range of values of x is $2x^2 - x - 3$ negative ?

7. Find the least value of $(x+1)(x+3)$ and the greatest value of $(x-1)(3-x)$. Illustrate by rough graphs.

8. Prove that $x^2 - 3x + 3$ is always positive. Interpret this graphically.

9. Find (i) the greatest value of $1 + 2x - x^2$;

(ii) the least value of $5x^2 - x$.

10. What is the least value of c, if $2x^2 - 12x + c$ is never negative ?

11. What is the greatest value of c, if $c + 3x - 2x^2$ is never greater than 2 ?

12. Find an integral value of x such that

$$5x - 1 < (x+1)^2 < 7x - 3.$$

13. If $4 + 2x - x^2 = k$, prove that $k \not> 5$.

14. (i) If $\dfrac{x+4}{(x+1)(x-8)} = k$ has real roots, what limit is there to the possible values of k ?

(ii) Prove that the value of $\dfrac{x+4}{(x+1)(x-8)}$ cannot lie between $-\frac{1}{3}$ and $-\frac{1}{27}$.

15. Find the limits within which the value of $\dfrac{3x^2+2}{2x^2-2x+1}$ must lie.

16. Prove that $\dfrac{x+1}{x^2-4}$ is capable of any value.

17. What can you say about m, if

$$x^2 - (m-1)x + (m-1)(m-2) = 0$$

has two real roots ?

18. What is the condition that $ax^2 + ax + c$ is always positive

19. Sketch graphs of the following functions :

(i) $3(x-2)^2$; (ii) $5(x-2)(4-x)$;

(iii) $-2(x-1)^2 - 6$; (iv) $-3(x+2)(x-1)$.

20. For what range of values of x are the following functions positive ? Sketch their graphs.

(i) $(x+1)(x-1)(x-3)$; (ii) $(2x-3)(x+2)(3x-1)$;

(iii) $(x^2-1)(x^2-9)$; (iv) $(2x-1)^2(3-x)$.

[*For a revision exercise on Ch. VIII-IX, see Appendix, Ex. W. 4, p.* 152].

TEST PAPERS, C. 21-35

C. 21

1. (i) Find the value of c if $x-2$ is a factor of x^3+cx^2-6x-8.
(ii) Solve $x^3+3x^2-6x=8$.

2. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that

$$\frac{a-e}{b-f}=\sqrt{\left\{\frac{2a^2-c^2+3e^2}{2b^2-d^2+3f^2}\right\}}.$$

3. If α, β are the roots of $2x^2-8x+5=0$, find the value of

(i) $\alpha^2+\beta^2$; (ii) $\frac{1}{\alpha}+\frac{1}{\beta}$; (iii) $\sqrt{\left(\frac{\alpha}{\beta}\right)}+\sqrt{\left(\frac{\beta}{\alpha}\right)}$.

4. (i) Simplify $\frac{(2x-y)^3+(x-2y)^3}{x^4+x^2y^2+y^4}$;
(ii) Solve for x, $b(x^2-1)=x(1-b^2)$.

5. What number must be added to each of the numbers 2, 6, 9 to obtain four numbers in proportion ?

C. 22

1. Find the numerical values of b and c if

$$x^4-4x^3-x^2+bx+c$$

(i) has x^2-3x+2 as a factor ; (ii) is a perfect square.

2. If a, b, c are in continued proportion, prove that

(i) $\frac{b^2-c^2}{a^2-c^2}=\frac{b^2}{a^2+b^2}$;

(ii) the fourth proportional to a, b, c is $\frac{b^3}{a^2}$.

3. Make x the subject of the formula $\frac{1+ax}{1+bx}=\frac{c}{d}$.
Simplify the result if $c=b$ and $d=a$.

4. (i) Construct an equation having as roots

(a) ± 3, 0 ; (b) $5\pm 2\sqrt{3}$.

(ii) What is the value of p if the equation $x^2-10x+p=0$ has (a) equal roots, (b) roots of the form a, $3a$?

5. Draw the graph of $y=1+3x-x^2$ for values of x from -1 to $+4$. Use your graph

(i) to solve $x^2=3x+1$;

(ii) to find the values of x between which $3x - x^2$ is greater than 1;

(iii) the values of c for which $1 + 3x - x^2 = c$ has no roots.

Sketch on the same figure the graph of $y = 3x - x^2$.

C. 23

1. (i) Find the numerical values of b and c if $x - 2$ is a common factor of $x^2 + bx - 10$ and $bx^2 - 5x + c$.

(ii) Express $\frac{7}{(x-2)(2x-1)}$ in the form $\frac{a}{x-2} + \frac{b}{2x-1}$.

2. (i) If $6x^2 = xy + 15y^2$, find the possible values of $\frac{x+y}{x-y}$.

(ii) If $10p = 15q = 6r$, find the 3 smallest positive integers to which p, q, r are proportional.

3. Solve for x and y,

(i) $2x + 4y = 7a - b$, $x - y = b - a$;

(ii) $x^3 - y^3 = x - y = 2$.

4. If $x^2 + 6x + 4 \equiv (x - \alpha)(x - \beta)$, find the values of

$$\text{(i) } \frac{1}{\alpha} + \frac{1}{\beta}; \quad \text{(ii) } \alpha^2 + \beta^2.$$

Also express $(x - 3\alpha)(x - 3\beta)$ as a quadratic function of x with numerical coefficients.

5. The rth term of a series is $2 \cdot (3^r) - \frac{1}{3}r$. What is the sum of n terms?

C. 24

1. (i) Divide $12x^4 + 7x^3 - 27x^2 - 4x + 6$ by $3x^2 - 2x - 2$.

(ii) Expand in ascending powers of x, as far as the term in x^3,

$$(1 - 2x + 3x^2 - 4x^3)^3.$$

2. Factorise (i) $x^3 - 5x^2 - 4x + 20$; (ii) $y^3 - 13y + 12$;

(iii) $z^3 - 27$.

3. If $\frac{a}{b} = \frac{c}{d}$, prove that $ab - cd$ is a mean proportional between $a^2 - c^2$ and $b^2 - d^2$.

4. Find the value of b if the values of x which satisfy the equation $b = 20x - 4x^2$ (i) are equal, (ii) differ by 3.

5. *Sketch* the graph of $y = 20x - 4x^2$ showing clearly where it cuts the x-axis. What facts about the graph are obtained from the answers to No. 4 ?

C. 25

1. (i) For what values of c has the expression

$$4x^2 + 4xy - 3y^2 + 2x + cy - 12$$

linear factors ? What are then the factors ?

(ii) Factorise $x^4 - x^3 - 11x^2 + 9x + 18$.

2. Solve for x, (i) $x^2 + 2qx = s^2 - q^2$;

(ii) $\dfrac{x^2+3x+5}{x^2-3x-5} = \dfrac{2x^2+x-3}{2x^2-x+3}$.

3. Simplify (i) $(a+b) \cdot \sqrt{\left(\dfrac{a-b}{a+b}\right)} - (a-b)\sqrt{\left(\dfrac{a+b}{a-b}\right)}$;

(ii) $\sqrt{(2\frac{1}{4} + \sqrt{5})}$.

4. If s is the sum of n terms of an A.P. whose 1st term is a and common difference is d, express a in terms of s, n, d.

Find an A.P. with common difference 4 and such that the sum of 10 terms is 230.

5. If α and β are the roots of $3x^2 - 2x - 4 = 0$, form the quadratic equation having as roots, α^2 and β^2.

C. 26

1. (i) Find the square root of $4a^4 - 4a^3 - 11a^2 + 6a + 9$.

(ii) Prove that $x^2 + 3x - 10$ is a factor of

$$x^4 + 2x^3 - 15x^2 + 4x + 20.$$

2. (i) Solve the simultaneous equations,

$$x(\sqrt{3}+\sqrt{2}) + y(\sqrt{3}-\sqrt{2}) = 6,$$
$$x(\sqrt{3}-\sqrt{2}) + y(\sqrt{3}+\sqrt{2}) = 6.$$

(ii) Solve for x, $(x+a)^2 = 2x^2$.

3. If $\dfrac{a}{y-z} = \dfrac{b}{z-x} = \dfrac{c}{x-y}$, prove that

(i) $a+b+c=0$; (ii) $x(b+c) + y(c+a) + z(a+b) = 0$.

4. (i) Find a quadratic equation with rational coefficients which has $2+\sqrt{5}$ as one of its roots.

(ii) Find the values of c if the equation, $(x+6)^2 = cx$, has (*a*) equal roots, (*b*) roots in the ratio 4 : 1.

5. A marble rolls down a groove with a gentle slope so that the distance travelled in t seconds is $\frac{1}{8}t^2$ feet. Find how far it goes in (a) the 1st second, (b) the 2nd second, (c) the 3rd second, (d) the rth second.

Show that the distances travelled in successive seconds form an A.P.

C. 27

1. (i) Simplify $\left(\frac{1}{a}-\frac{1}{b}\right)(a^2+ab+b^2)\div\left(\frac{a}{b^2}-\frac{b}{a^2}\right)$.

   (ii) Divide y^5+y+1 by y^2+y+1.

2. Prove that $a-b$ is a factor of

$$(a+b)^2(a-b)+(b+c)^2(b-c)+(c+a)^2(c-a).$$

What are the other factors ? Hence factorise the expression.

3. (i) If $2x-y+3z=0$ and $3x+2y-z=0$, find $x:y:z$.

   (ii) If $\frac{x+y}{a}=\frac{x-z}{b}=\frac{x+y-z}{c}$, express $x:y:z$ in terms of a, b, c.

4. Make a table of values from $x=-2$ to $x=8$ for the function $y=4+\frac{12}{x-3}$, and show where there are missing values, and draw the corresponding graph.

Solve graphically the simultaneous equations,

$$(y-4)(x-3)=12, \quad y=2x+1.$$

5. A stone, thrown vertically upwards, is h feet above the ground after t seconds, where $h=40t-16t^2$.

What can you say about the numerical value of h if this equation has no roots, and what does this mean ?

After what time does the stone strike the ground ?

C. 28

1. If a, b, c are in continued proportion, express $\frac{b(a-c)}{a-b}$ in terms of b, c.

2. Find the values of p and q if $(x-1)^2$ is a factor of

$$x^3+4x^2-(2p-q)x+pq.$$

3. (i) Solve $8xyz=20(y+z)=15(z+x)=30(x+y)$.

   (ii) Solve for x, $\frac{a-b}{x-b}+\frac{a-c}{x-c}=2$.

4. Make A the subject of

(i) $\log A = \log P + n \log R$; (ii) $\log A = \frac{1}{3}\log Q - \frac{1}{4}\log R$

Find to the nearest whole number the value of n from the formula $A = P\left(1 + \frac{r}{100}\right)^n$, if $r = 3$, $P = 4$, $A = 10$.

5. If α, β are the roots of the equation $3x^2 - 7x + 3 = 0$,

(i) find the values of $\frac{1}{\alpha + 1} + \frac{1}{\beta + 1}$;

(ii) form the equation whose roots are 2α and 2β;

(iii) form the equation whose roots are $\alpha + 1$ and $\beta + 1$.

C. 29

1. Prove that $x = 3$ is a root of the equation,

$$x^3 + x^2 - 16x + 12 = 0.$$

Find the other roots correct to 2 places of decimals.

2. (i) Find the term independent of x in the expansion of $\left(2x - \frac{3}{x^2}\right)^9$.

(ii) What is the coefficient of y^9 in $(1 - 2y)(1 + y)^{10}$?

3. Prove that

$$(x + y - z)^3 + (y + z - x)^3 + (z + x - y)^3 - (x + y + z)^3$$

reduces to $cxyz$, where c is a constant and finds its value.

4. Draw the graph of $y = x^3 - 3x + 3$ for values of x from -2 to $+2$. From the graph, find

(i) the least value of y for positive values of x;

(ii) the greatest value of y for negative values of x;

(iii) the roots of $x^3 - 3x = 1$.

Draw as much more of the graph as is necessary for solving $x^3 - 3x = 4$, and then solve it.

5. The Horse-Power at which a man works in climbing a mountain varies directly as his weight and the height climbed and inversely as the time taken. Compare the rate of working of two men, one of whom weighs 10 stone and climbs 1000 feet in 50 minutes, while the other weighs 12 stone and climbs 1100 feet in an hour.

C. 30

(i) If $(a+b):(a-b)=c:d$, prove that $(c+d):(c-d)=a:b$.

(ii) Make y the subject of the formula

$$x=\log_{10}(24-2y)-\log_{10}24.$$

2. Sum to 10 terms the series,

$$2+\sqrt{2}+1+\frac{1}{\sqrt{2}}+\dots.$$

By how much does this sum fall short of the sum to infinity?

3. (i) Express $7x-11$ in the form $b(x-2)+c(x-3)$ where b, c are constants.

(ii) If $ax^5+bx^4+cx^2+dx+e$ is divided by $x-2$, the remainder is 1; also if it is divided by $x-3$, the remainder is 2. Show that the remainder is $x-1$ if it is divided by $(x-2)(x-3)$.

4. (i) Prove that the roots of $2x^2=5x+11$ differ by the same amount as do the roots of $2x^2+7x=8$.

(ii) Express c in terms of b if the roots of $x^2+2bx=c$ differ by $6b$.

5. For what range of values of x is $4+5x-6x^2$ positive? Express $4+5x-6x^2$ in the form $a-b(x-c)^2$; hence find the greatest value of $4+5x-6x^2$ and the value of x for which it is greatest.

C. 31

1. If $x^{\frac{1}{2}}y^{-\frac{3}{4}}z^{-\frac{2}{3}}=5$, express y in the form cx^pz^q, giving c correct to 2 figures.

2. (i) Find the value of b if $x-2$ is a factor of

$$2x^3-11x^2+bx-6,$$

and in this case find the other factors.

(ii) Factorise

$$(a^2-b^2)(a+b)^2+(b^2-c^2)(b+c)^2+(c^2-a^2)(c+a)^2.$$

3. (i) Solve for x, y, z,

$$x:y:z=3:2:4;\ 4x-2y-z=6.$$

(ii) What is the sum of the cubes of the roots of

$$\frac{1}{x}+\frac{1}{x+1}=3?$$

4. (i) Prove that the sum of the odd numbers from 1 to 71 inclusive is equal to the sum of the odd numbers from 97 to 119 inclusive.

(ii) If the sum of n consecutive odd numbers is equal to the sum of the first $3n$ odd numbers, find the greatest and least of the former set.

5. By plotting $\log y$ against $\log x$, find a simple relation, giving y as a function of x, from the following data:

y	0·2	1·6	5·4	12·8	25
x	1	2	3	4	5

C. 32

1. If $x=a-b$, simplify the expression

$$\frac{1}{x^2+a^2-b^2}-\frac{1}{x^2-a^2+b^2}+\frac{1}{x^2-a^2-b^2}.$$

2. Find a, b if $x^4-bx^3-11x^2+4(b+1)x+a$ is a perfect square and if $x+2$ is a factor.

3. (i) If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $\frac{a-c}{b-d}=\sqrt[3]{\left(\frac{a}{d}\right)}$.

(ii) If $\frac{a}{x}+\frac{b}{y}=\frac{a}{y}$, express $\frac{x+y}{x-y}$ in terms of a, b.

4. (i) Solve $x^3-3x^2+20=4(x+2)$.

(ii) If $xy=c^2$, $yz=a^2$, $zx=b^2$ and if $x^2+y^2+z^2=d^2$, find an equation connecting a, b, c, d.

5. The weight y gm. which produces a given deflexion in the middle of a beam supported at two points x cm. apart, equidistant from the ends is determined experimentally, as follows:

x	50	60	70	80
y	300	174	109	73

Show graphically that x and y are connected by a relation of the form $x^ny=c$, and find the best-fit values of n, c.

C. 33

1. (i) Find n correct to 3 figures if $7^{n+1}=10^n$.

(ii) Find y in terms of x if $\log_{10} y=0·75\log_{10} x+0·82$.

2. (i) Evaluate $\dfrac{3^{0\cdot8}}{27^{-0\cdot4}} - \dfrac{2^{-1}}{4^{-1\cdot5}}$.

(ii) Find the product of the four expressions, $\sqrt{a} \pm \sqrt{b} \pm \sqrt{c}$.

3. Solve (i) $3x^2 + 5xy - 2y^2 = 5$, $5x - 2y = 1$.

(ii) $z^{\frac{1}{3}} - 10z^{-\frac{1}{3}} = 3$.

4. If α, β are the roots of $3x^2 + x - 7 = 0$, prove that

$$(3\alpha + 1)^2 + (3\beta + 1)^2 + 9(\alpha + \beta) = 40.$$

5. Find the product of n terms of a G.P. if

(i) the 1st term is a and the common ratio is r,

(ii) the 1st term is a and the last term is l,

(iii) the 1st term is a and the 2nd term is b.

C. 34

1. (i) If $x^4 + 2x^3 + 3x^2 + 4x + 5 \equiv (x-1)(x-2)(x^2 + px + q) + ax + b$, find numerical values of a and b.

(ii) What is the remainder if $x^4 + 2x^3 + 3x^2 + 4x + 5$ is divided by $(x-1)(x-2)$?

2. (i) Make x the subject of the formula $\left(\dfrac{a}{b}\right)^{\frac{1+x}{x}} = c$.

(ii) If $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d}$, express $\dfrac{(a+b)^3}{(b+c)^3}$ in terms of a, d.

3. (i) If a, b, c are in G.P., prove that $a+b$, $2b$, $b+c$ are in H.P.

(ii) If a, x, b are in A.P. and if a, y, b are in H.P., find the geometric mean of x and y in terms of a, b.

4. If α, β are the roots of $px^2 + qx + r = 0$, find in terms of the coefficients,

(i) $(q\alpha + r)(q\beta + r)$; (ii) $(p\alpha + q)(p\beta + q)$.

5. The following swimming records are taken from *Whitaker's Almanack*:

Distance in yards	- d	150	220	300	440	500
Time in seconds	- t	92·4	145·4	210	323	367·2

Plot $\log t$ against $\log d$, and so find a possible formula for t in terms of d. The record for 300 metres is 230·2 sec. Does this agree with the formula you have found? [1 metre = 1·094 yards].

C. 35

1. (i) Express
$$\frac{x(7x-13)}{(x-1)^2(x+2)} \text{ in the form } \frac{a}{(x-1)^2}+\frac{b}{x-1}+\frac{c}{x+2},$$
where a, b, c are constants.

(ii) Simplify $\dfrac{\sqrt{(1+\sqrt{5})}}{\sqrt{(2+\sqrt{5})}}$.

2. (i) Find the possible values of c if $x+2$ is a factor of
$$x^3+4cx^2+(c+1)^2x-6.$$

(ii) Show that x is one factor and that $(y+z)$ is another factor of $(yz+zx+xy)^3-y^3z^3-z^3x^3-x^3y^3$. Hence factorise this expression.

3. Evaluate (i) $(\sqrt{3}+\sqrt{2})^6+(\sqrt{3}-\sqrt{2})^6$;

(ii) $(\sqrt{3}+\sqrt{2})^{-6}+(\sqrt{3}-\sqrt{2})^{-6}$.

4. (i) Sum the series $1+(3+\frac{1}{3})+\left(3^2+\dfrac{1}{3^2}\right)+\ldots+\left(3^n+\dfrac{1}{3^n}\right)$.

(ii) Find the value of the nth bracket and the sum of the first n brackets of the following series:

$(1+2)$, $(3+4+5+6)$, $(7+8+9+10+11+12)$, ...

5. The graph of $y=2x^2+bx+24$ cuts the x-axis at P, Q and the y-axis at R. If the area of $\triangle$PQR is 78 units of area, find b.

[*For additional test papers, see Appendix, Z.* 21-30, *p.* 193.]

APPENDIX

REVISION EXERCISE W. 1. (Ch. I-II)

1. (i) Multiply c^3 by $(c^3)^2$; (ii) cube $4d^4$.

2. Divide (i) $a^{15} \times a^{10}$ by a^5 ; (ii) $(b^2)^5$ by b^2.

3. What are the values of 6^{-2}, 5^{-1}, $4^{-\frac{1}{2}}$, 3^0 ?

4. Evaluate to 3 figures (i) $\dfrac{18 \cdot 63 \times 94}{7 \cdot 07}$; (ii) $\dfrac{\sqrt{10}}{\sqrt{6}}$.

5. Find n if (i) $y^n \times y^n = 1$; (ii) $y^n \times y^n = \dfrac{1}{y}$.

6. Find to 3 figures $\pi r^2 h$ if $r = 1 \cdot 08$, $h = 0 \cdot 85$.

7. Divide the cube of $2p^2$ by the square of $2p^3$.

8. What are the values of $4^{-\frac{3}{2}}$, $16^{-\frac{3}{4}}$, $100^{2 \cdot 5}$, $1^{-\frac{1}{2}}$?

9. Evaluate to 3 figures (i) $4 \cdot 837 \times \sqrt{7 \cdot 219}$; (ii) $\dfrac{1}{(0 \cdot 273)^2}$.

10. If y varies inversely as x, and if $y = 3$ when $x = 5$, find y when $x = 2$.

11. Express without using fractional or negative indices :

$$\text{(i) } y = ax^{-2} \text{ ; (ii) } pv^{-1} = c \text{ ; (iii) } z = x^{-\frac{1}{2}} y^{\frac{3}{2}}.$$

12. If $x = 16$, $y = 9$, write down the values of

$$\text{(i) } x^{\frac{1}{2}} + y^{\frac{1}{2}} \text{ ; (ii) } (x + y)^{\frac{1}{2}} \text{ ; (iii) } \left(\frac{x}{y}\right)^{-\frac{1}{2}}.$$

13. Evaluate to 3 figures (i) $\sqrt{\dfrac{836 \cdot 4}{92 \cdot 07}}$; (ii) $\sqrt[3]{\dfrac{1}{0 \cdot 7564}}$.

14. A cube of aluminium, edge x cm., weighs y gm. What variation-relation connects x and y ? If $y = 325$ when $x = 5$, find y when $x = 4$.

15. Express as a power of 4, (i) 8 ; (ii) $\frac{1}{4}$; (iii) 1.

16. Evaluate to 3 figures $a^2 b^3 c^{-\frac{1}{2}}$ when $a = 18 \cdot 37$, $b = 0 \cdot 0715$, $c = 0 \cdot 008472$.

17. Simplify (i) $(a^3 b^{\frac{1}{2}})^{\frac{1}{2}}$; (ii) $c^3(c^{-1} + c^{-2})$.

18. If $V = \frac{1}{3}\pi r^2 h$, find r when $V = 203$, $h = 10 \cdot 7$.

19. Complete the following: (i) If $s \propto t^3$, then $t \propto \ldots$; (ii) if $r \propto \sqrt[3]{V}$, then $V \propto \ldots$.

20. Find n if (i) $3^n = 9$; (ii) $9^n = 3$; (iii) $(\frac{1}{3})^n = \sqrt{3}$.

21. Evaluate (i) $0{\cdot}0847 \times \sqrt[5]{0{\cdot}0716}$; (ii) $\sqrt[4]{\dfrac{1}{0{\cdot}0307}}$.

22. The pressure, P lb. per sq. ft., on the wind-screen of a car travelling v miles per hour, obeys the law, $P \propto v^2$. If $P = 2$ when $v = 20$, find (i) P if $v = 30$, (ii) v if $P = \frac{1}{2}$.

23. Find approximately the value of (i) 2^{30}; (ii) 3^{100}.

24. If $x \propto \sqrt{y}$ and $y \propto \dfrac{1}{z}$, what is the effect on x of multiplying z by 100? What is the effect on z of multiplying x by 100?

25. If $R = \dfrac{4l\rho}{\pi d^2}$, find R when $\rho = 1{\cdot}7 \times 10^{-6}$, $l = 94300$, $d = 0{\cdot}25$.

26. Express as a power of 64, (i) 8; (ii) 2; (iii) 32; (iv) $\frac{1}{4}$; (v) 256.

27. If p varies directly as t and inversely as v, and if $p = 240$ when $t = 400$ and $v = 60$, find p in terms of t, v.

28. If $d^3 = \left(\dfrac{c}{12{\cdot}8}\right)^2$, find d when $c = 72{\cdot}5$.

29. Evaluate, without using tables,

(i) $\log_{10} 10^{3\cdot5}$; (ii) $100^{1\cdot5}$; (iii) $\log_{10} 0{\cdot}01$.

30. If $x \propto \dfrac{1}{y}$ and if $x \propto z^2$, what is the effect on y of doubling z?

31. In what ratio must the volume of a sphere be increased in order to double the surface?

32. Evaluate $a^{\frac{2}{3}}b^{-1}c^{-3}$ when $a = 0{\cdot}0376$, $b = 16{\cdot}92$, $c = 0{\cdot}3947$.

33. If $\frac{4}{3}\pi r^3 = 10$, find to 3 figures the value of $4\pi r^2$.

34. The volume of a regular octahedron of edge 2 in. is 3·77 cu. in. What is the edge of a regular octahedron whose volume is 5 cu. in.?

35. Evaluate (i) $50 \times (1{\cdot}05)^6$; (ii) $50 \times (1{\cdot}05)^{-6}$.

36. If z varies as the square of x and inversely as the square root of y, and if $z = 8$ when $x = 6$ and $y = 2\frac{1}{4}$, find y in terms of x, z.

37. Evaluate $x^{\frac{2}{3}} \div \sqrt{(y^2 + z^2)}$ when $x = 1{\cdot}037$, $y = 0{\cdot}208$, $z = 0{\cdot}396$.

38. If $V \propto r^3$ and if $r \propto \dfrac{1}{t^2}$, what is the effect on V of halving t?

39. If $f = 1{\cdot}925 \times 10^{-3}q^2d^4$, find d when $f = 11{\cdot}03$, $q = 17{\cdot}94$.

40. Given that $\log_{10} 2 = 0{\cdot}30103$, find $\log_{10} 2{\cdot}5$ without using tables.

41. The force necessary to stop a train in a given distance varies directly as the weight of the train and the square of its speed and inversely as the distance. A force of 10 tons will stop a train weighing 200 tons travelling at 30 m.p.h. in 200 yd.; find the formula for the force F tons required to stop a train, weight W tons, travelling at v m.p.h. in d yards.

42. If $\pi r^2 = 43{\cdot}7$, evaluate to 2 figures $r^3 - \dfrac{14{\cdot}64}{r}$.

43. If $z \propto \dfrac{x^2}{y}$ and if $x \propto \dfrac{w}{r^3}$ and $y \propto \dfrac{r^2}{w^2}$, find (i) how z varies with w and r; (ii) how r varies with x, z.

44. If $z = 3{\cdot}58x^{0{\cdot}8}y^{-0{\cdot}4}$, express x in the form, $x = cy^p z^q$, giving c correct to 3 figures.

REVISION EXERCISE W. 2. (Ch. III-IV)

1. Simplify (i) $\sqrt{15} \times \sqrt{60}$; (ii) $\sqrt{72} \div \sqrt{200}$.

2. Simplify (i) $\log x^3 \div \log\left(\dfrac{1}{x}\right)$; (ii) $\log y^3 - \log\left(\dfrac{1}{y^3}\right)$.

3. Without using tables, state which is the greater:
   (i) $\log 1 + \log 3$ or $\log(1+3)$;
   (ii) $\log 2 + \log 4$ or $\log(2+4)$;
   (iii) $\log 2 - \log 1$ or $\log(2-1)$;
   (iv) $\log 6 - \log 3$ or $\log(6-3)$.

4. Evaluate $4a - a^2$ if $a = 2 - \sqrt{3}$.

5. (i) Multiply $\sqrt[3]{x}$ by $\sqrt[5]{x}$; (ii) Divide $\sqrt[3]{y^2}$ by $\sqrt[5]{y}$.

6. Simplify (i) $\dfrac{12}{\sqrt{2}}$; (ii) $\dfrac{\sqrt{18}}{\sqrt{2}}$; (iii) $\sqrt{12} - \sqrt{3}$.

7. Find, without using tables, the values of
   (i) $\log_2 16$; (ii) $\log_2(\frac{1}{32})$; (iii) $\log_2(2\sqrt{2})$; (iv) $\log_2\left(\dfrac{1}{\sqrt{2}}\right)$.

8. Expand (i) $(2\sqrt{3} - \sqrt{6})^2$;
   (ii) $(\sqrt{3} + \sqrt{5} - 2\sqrt{2})(\sqrt{3} + \sqrt{5} + 2\sqrt{2})$.

9. What can you say about x if $(0{\cdot}7)^x < 0{\cdot}001$?

10. Simplify (i) $\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$; (ii) $\dfrac{1}{\sqrt{18} - \sqrt{8}}$.

11. Find x, y if $\dfrac{\log x}{\log 5} = \dfrac{\log 36}{\log 6} = \dfrac{\log 64}{\log y}$.

12. Find the square root of $21 - 6\sqrt{10}$.

13. If $\log_{10} 12 = a$ and $\log_{10} 18 = b$, find in terms of a, b the values of $\log_{10} x$ for $x = 2, 3, 4, 5, 6, 8, 9$.

14. Solve the equation, $\sqrt{x} + \sqrt{(3x+1)} = 3$.

15. Simplify (i) $10^{\log_{10} 3}$; (ii) $\dfrac{\log 27}{\log 9}$; (iii) $\dfrac{\log 32}{\log 16}$.

16. Evaluate $a + \dfrac{1}{a}$ if $a = \dfrac{7 - 3\sqrt{5}}{7 + 3\sqrt{5}}$.

17. Express in a form not involving logarithmic notation the following, where the logarithmic base is 10.

(i) $\log p + \log q = 2$;	(ii) $\log r = 1 + \log s$;
(iii) $2 \log x - 3 \log y = 1$;	(iv) $a \log 2 = b \log 5$;
(v) $b = 3 \log c + 1$;	(vi) $x = 10^{3 \log y}$.

18. Simplify $\dfrac{a - b}{a\sqrt{b} - b\sqrt{a}}$.

19. If $8^x = 16^y$, find the value of $\dfrac{x}{y}$.

20. If x is small compared with 1, $\log(1 + x) \simeq 0{\cdot}434x$; use this fact to find approximately a root of $\log(1 + x) = \dfrac{\sqrt{x}}{10}$.

21. If $a = 10^x$, $b = 10^y$, $c = 10^z$, express $\log_{10}\left(\dfrac{10\sqrt{b}}{a^3 c}\right)$ in terms of x, y, z.

22. Find, without using tables, the values of

$$\log_8 x \text{ for } x = 2,\ 4,\ \sqrt[3]{2},\ \tfrac{1}{16}.$$

23. Find the square root of $17 - 4\sqrt{15}$.

24. Find x if $\log_e x = 1{\cdot}5$ when $e = 2{\cdot}718$.

25. Evaluate (i) $\log_5 3$; (ii) $\log_3 5$.

26. If $\log_{10} V = 2 \log_{10} r + \log_{10} h + 0{\cdot}497$, express V in terms of r, h.

27. Find x if $\log_8 x = y$, for $y = 2\tfrac{1}{2}, -1\tfrac{1}{3}, 0$.

28. Find x if $4^{x+1} = 40$.

29. If $\log_a p - \log_a(\sqrt{q}) = 2$, express q in terms of p, a.

30. If $y = ax^n$ is satisfied by $x = 2$, $y = 10{\cdot}6$ and by $x = 3$, $y = 6{\cdot}2$, find n and a.

31. Solve $\sqrt{(7 + x)} - \sqrt{(x - 1)} = \sqrt{(6 - x)}$.

32. Simplify $\log_a(a^2 x) - \log_b(b^2 x)$.

33. Find x, y if $\log_6 x + \log_6 y = 2$ and $\log_2 x - \log_2 y = 6$.

34. Simplify $\sqrt{(\sqrt{11} - \sqrt{10})} \div \sqrt{(\sqrt{11} + \sqrt{10})}$.

35. Solve $25^x = 5^{x+1} - 6$.

36. If $\dfrac{1}{4b^2} = \dfrac{2b}{(a^2+b^2)^{\frac{3}{2}}}$, find $\dfrac{a}{b}$.

37. Find x if $2^{3-x} = 3^{2-x}$.

38. Simplify $\dfrac{\log_a x}{\log_b x} - \dfrac{\log_a y}{\log_b y}$.

39. If $y = \log_{10}(\log_{10} x)$, find y when $x = 10$ and find x when $y = 1$.

40. If $2^x \,.\, 3^y = 3^x \,.\, 4^y = 6$, prove that $x^2 - 2y^2 = 2x - 3y$.

REVISION EXERCISE W. 3. (Ch. V-VI)

1. The nth term of a series is $n^2 + n$, write down the first 4 terms. What is the $(2n)$th term ?

2. If $\dfrac{a}{b} = \dfrac{7}{3}$, evaluate (i) $\dfrac{a+b}{a-b}$; (ii) $\dfrac{a^2+b^2}{a^2-b^2}$.

3. How many terms are there in the following series in A.P. ?
(i) 31, 38, ... , 185; (ii) 143, 140, ... , 86; (iii) $x, y, \ldots, z$.

4. Write down the nth term of a G.P. if the first two terms are (i) $1, a$; (ii) $b, 1$; (iii) x^3, x^6; (iv) $1, -2$; (v) 48, 72.

5. If $x : y : z = \frac{1}{2} : \frac{1}{4} : \frac{1}{5}$, evaluate (i) $\dfrac{x+y}{y+z}$; (ii) $\dfrac{x^2-y^2+z^2}{(x-y)(y-z)}$.

6. What is the least number above 100 which belongs to the series, (i) 3, 14, 25, 36, ... ; (ii) $\frac{500}{50}, \frac{501}{49}, \frac{502}{48}, \frac{503}{47}, \ldots$?

7. Simplify (i) $16(-\frac{1}{2})^7$; (ii) $144 \,.\, (\frac{3}{2})^{n-1}$.

8. Find a possible form for the nth term in the series,

(i) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots$; (ii) 10, 200, 3000, 40000, ... ;

(iii) $\frac{1}{2}, \frac{4}{5}, \frac{9}{8}, \frac{16}{11}, \frac{25}{14}, \ldots$; (iv) $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \ldots$;

(v) 0·1, 0·02, 0·003, 0·0004, ... ;

(vi) 4, −7, 10, −13, 16, −19, ...

9. If $\dfrac{a}{b} = \dfrac{c}{d}$, prove that $\dfrac{ac+bd}{ac-bd} = \dfrac{a^2+b^2}{a^2-b^2}$.

Sum the series,

10. 11, 12, 13, 14, ... , 29, 30;

11. 26, 23, 20, 17, ... (20 terms).

12. What is the arithmetic mean of

(i) $\frac{2}{3}, -\frac{1}{6}$; (ii) $(a+b)^2, (a-b)^2$; (iii) $\log p, \log q$?

13. If $3x - y = 2z$ and $y = 6(z - x)$, find $x : y : z$.

14. The nth term of a series is $2 . 10^{n-1}+5n$; write down the first 3 terms.

15. Sum the following series in G.P., leaving the answer in the index form,

(i) $\frac{3}{8}, \frac{1}{4}, \frac{1}{6}, \frac{1}{9}, \ldots$ 10 terms; (ii) $8, -16, 32, -64, \ldots$ 12 terms.

16. The nth term of a series is $5n-7$; what are the first 3 terms? Find also the sum of the first k terms.

17. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that $\frac{ac-e^2}{bd-f^2}=\frac{ae}{bf}$.

18. Sum the following series in A.P.:

(i) $4{\cdot}2, 6{\cdot}1, 8, 9{\cdot}9, 11{\cdot}8, \ldots$ (40 terms);

(ii) $14, 16\frac{1}{2}, 19, \ldots, 39$.

19. What number must be added to each of the numbers, 7, 12, 16, 24 to make them in proportion?

20. Using logarithms, find an approximate value of

(i) $40+60+90+135+\ldots$ to 12 terms;

(ii) $20+20(1{\cdot}04)+20(1{\cdot}04)^2+\ldots$ to 15 terms.

21. The 3rd and 8th terms of an H.P. are $\frac{1}{11}$ and $\frac{1}{26}$; what is the first term?

22. If $\frac{x+y}{6}=\frac{y+z}{7}=\frac{z+x}{8}$, find $x:y:z$.

23. If n is an integer, find the sum of all the integers *between* n and $2n$.

24. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $\frac{a-b}{b-c}=\frac{b-c}{c-d}$.

25. The first and last terms of an A.P. are 19 and 187; find the number of terms if the common difference is

(i) 2; (ii) 0·2; (iii) 7.

26. If $x:y:z=6:9:10$, find three integers to which $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are proportional.

27. What is the sum of all numbers divisible by 12 between 2000 and 5000?

28. The perimeter of a right-angled triangle is 4 times the shortest side; find the ratio of the other two sides.

29. What is the mean proportional between

(i) a^3b and ab^3; (ii) $\frac{c}{d}$ and $\frac{d}{c}$?

30. In an A.P. the last term is l, the last term but one is $\frac{9l}{10}$, and there are n terms. What is the first term? What is the sum of the n terms?

31. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, complete the relation $\frac{a^3-c^3}{(a+e)^3}=\frac{\dots}{\dots}$.

32. Sum to 8 terms the G.P., $81, -36, 16, -\frac{64}{9}, \dots$.

33. If $6(x^2-y^2)=5xy$, find the value of $\frac{x+2y}{2x-y}$.

34. Divide $\{1+3+5+\dots+(2n-1)\}$ by

$$\{(2n+1)+(2n+3)+\dots+(4n-1)\}.$$

35. If $\frac{a}{5}=\frac{b+c}{11}=\frac{a-b+c}{2}$, find $\frac{1}{a}:\frac{1}{b}:\frac{1}{c}$.

36. Find the limiting sum of $1000+800+640+512+\dots$.

37. If a, b, c are in A.P. and if b, c, d are in H.P., prove that $ad=bc$.

38. The sum of the first n terms of a series is $\frac{n(n+1)(2n+1)}{6}$, for all positive integral values of n. What are the first 3 terms? What is the rth term?

39. If $ax^2+bxy+cy^2=0$ and $px^2+qxy+ry^2=0$, express in terms of a, b, c, p, q, r in two forms, without radical signs, the ratio $\frac{x}{y}$.

40. Find the limiting sum of the series $30-6+1{\cdot}2-0{\cdot}24+\dots$.

41. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that

$$(a+b+c+d)^2=(a+b)^2+2(b+c)^2+(c+d)^2.$$

42. In the series,

$$[1],\ [3+5],\ [7+9+11],\ [13+15+17+19],\ \dots$$

find (i) the first number in the nth bracket, (ii) the sum of the numbers in the nth bracket, (iii) the sum of the first n brackets.

43. If $\frac{y+z-x}{a}=\frac{z+x-y}{b}=\frac{x+y-z}{c}$, express the ratios

$$x(y-z):y(z-x):z(x-y)$$

in terms of a, b, c.

44. A pendulum is set swinging; its first vibration is through 30°, and each succeeding vibration is $\frac{4}{5}$ of the one before it. What is the total angle through which it swings before coming to rest?

REVISION EXERCISE W. 4. (Ch. VIII-IX)

1. Find x if (i) $x+cx=c^2-1$; (ii) $(x+b)^2=(x-b)(x+2b)$.

2. Find x, y if $x+by=ab$, $bx+ay=a^2$.

3. Find x, y if $ax-by=0$, $a^2x+b^2y=c$.

4. Find x, y if $y+sx=\dfrac{a}{s}$, $y+tx=\dfrac{a}{t}$.

5. Find x if $x^2-2px=a$.

6. Find x if $x^2+2ax=1-a^2$.

7. If $2s=a+b+c$, prove that $(s-a)^2+ac=(s-b)^2+bc$.

Prove the following identities, Nos. 8-10.

8. $x^2(y-z)+y^2(z-x)+z^2(x-y)\equiv-(x-y)(y-z)(z-x)$.

9. $(x-y)^2+(y-z)^2+2(x-y)(y-z)\equiv(x-z)^2$.

10. $(a-b)(a^2+ab+b^2)+(b-c)(b^2+bc+c^2)\equiv(a-c)(a^2+ac+c^2)$.

11. If $a+b+c=0$, prove that $a^2-bc=b^2-ca$.

12. O is the mid-point of a line AB; AB is produced to P so that AP . BP $=$ AB^2. If AB $=2x$ in. and BP $=y$ in., find a relation between x and y. Hence prove that OP$^2=5$OA^2.

13. Solve the equation, $x^3=39x+70$.

14. Find the values of a, b if $x-2$ is a common factor of $x^2+ax-a-b$ and $x^2-2ax+b+1$.

15. Find the factors of (i) $2x^6-16$; (ii) a^4+4b^4.

16. Find the factors of $(2a+b)^3+(a+2b)^3$.

17. Simplify $\left(\dfrac{1}{b^2}-\dfrac{1}{c^2}\right)\left(b+\dfrac{c^2}{b}\right)\div\left(\dfrac{b}{c^2}-\dfrac{c^2}{b^3}\right)$.

18. Simplify $\dfrac{\dfrac{x}{y}+\dfrac{y}{x}-1}{\dfrac{x}{y}-\dfrac{y}{x}}\div\dfrac{\dfrac{1}{x^3}+\dfrac{1}{y^3}}{\left(\dfrac{1}{x}-\dfrac{1}{y}\right)^2}$.

19. Simplify $\dfrac{x^3-1}{x^2+2x-3}-\dfrac{x^3-8}{x^2+x-6}$.

20. Find the square root of $9x^4-12x^3-8x^2+8x+4$.

21. Factorise $9a^2-9ab-10b^2+6a-17b-3$.

22. For what values of b, c is x^2-2x+5 a factor of
$$2x^4-3x^3+bx^2+cx-15\ ?$$

23. Find the values of b, c if $8x^3-60x^2+bx+c$ is a perfect cube.

24. Express $\dfrac{3x+11}{(x-3)(x+1)}$ in the form, $\dfrac{a}{x-3}+\dfrac{b}{x+1}$.

25. Express x^2 in the form,

$$a(x-1)(x-2)+b(x-2)(x-3)+c(x-3)(x-1),$$

where a, b, c are independent of x.

26. Show that $axy+bx+cy+d$ has factors if $bc=ad$.

27. Simplify $\dfrac{(x+a)(x-b)}{(ax+c^2)(bx-c^2)}$ if $x=\dfrac{ab-c^2}{a-b}$.

28. If b, c are consecutive integers, b being the smaller, prove that $c^3-b^3=3bc+1$.

29. Solve $ax-(a-b)y=b^2$, $(a+b)x-by=a^2-b^2$.

30. Solve $2x+y=3z+4y-x=2y+z+2$, $y-z=6$.

31. Find the value of c if $x+1$ is a factor of

$$cx^3-(c-1)x^2-(2c+1)x-c+1\,;$$

find in this case the other factors.

32. Find the values of a, b if $x-1$ is a common factor of

$$2x^2-(a-1)x-b \text{ and } 3x^2-(2a+1)x+2b.$$

33. If $2(x+1)$ and $3(x-4)$ are the squares of consecutive integers, find the value of x.

34. Solve $ax-by=1$, $ax^2-by^2=\dfrac{1}{a-b}$.

35. If $a+b=3$, prove that $a^3+b^3+9ab=27$.

36. For what value of x has $\dfrac{x^2+a^2}{x-a}$ the same value as it has when $x=b$?

37. If $x=2$ and $x=-1$ are two roots of the equation,

$$ax^3-3x^2+bx+2=0,$$

find the third root.

38. Express $2x^2-3x-5$ in the form, $a(x-1)(x-2)+b(x-1)+c$.

39. If $a+\dfrac{1}{b}=b+\dfrac{1}{c}=1$, prove that $c+\dfrac{1}{a}=1$.

40. If $x=\dfrac{a+y}{b-1}=\dfrac{a-y}{b+1}$, find x in terms of a, b.

41. Factorise $(x-1)(x-2)(x-3)+24$.

42. If $a=x^2-yz$, $b=y^2-zx$, $c=z^2-xy$, prove that $\dfrac{ac-b^2}{ab-c^2}=\dfrac{y}{z}$.

43. Prove that $\dfrac{1}{x-a}+\dfrac{1}{x-b}$ has the same value when $x=a+b$ as it has when $x=\dfrac{2ab}{a+b}$.

44. What is the coefficient of x^3 in

(i) $(3-4x)^{10}$; (ii) $(1-2x)(1+5x)^6$?

45. What is the ratio of the $(r+1)$th term in $(1+x)^n$ to the rth term ? What can you say about r if the $(r+1)$th term is greater than the rth term ?

EXERCISE F.P. 1

Indices. (Chapter I)

1. Find n if (i) $x^{18} \div x^6 = x^n$; (ii) $x^4 \div x^{12} = x^n$;
(iii) $x^3 \div x^n = x^{-6}$; (iv) $x^3 \div x^3 = x^n$.

2. Find n if (i) $x^n \times x^5 = x^4$; (ii) $x^n \times x^6 = x^6$;
(iii) $x^n \times x^n = x^{16}$; (iv) $x^n \times x^n = x$.

3. Express as powers of x, *i.e.* in the form x^n,

(i) $\dfrac{x^{12}}{x^3}$; (ii) $\dfrac{x^3}{x^{12}}$; (iii) $\dfrac{x^3}{x^3}$; (iv) $\dfrac{1}{x^4}$; (v) 1.

4. Express with positive indices :

(i) a^{-2}; (ii) b^{-1}; (iii) $\left(\dfrac{1}{c}\right)^{-3}$; (iv) $d^{-1} \times d^{-1}$.

5. What are the values of :

(i) 4^{-2}; (ii) 3^{-3}; (iii) 5^{-1}; (iv) 6^0;
(v) $(\frac{1}{3})^{-2}$; (vi) $(\frac{2}{3})^{-3}$; (vii) $(0{\cdot}1)^{-2}$; (viii) $(\frac{4}{5})^{-1}$.

6. If $a=2$, $b=3$, write down the values of

(i) $a^{-1}+b^{-1}$; (ii) $(a+b)^{-1}$; (iii) ab^{-1}; (iv) $(a-b)^{-1}$.

7. Assign a meaning, with reasons, to (i) x^{-2}; (ii) $y^{\frac{1}{2}}$.

8. What are the values of :

(i) $16^{\frac{1}{2}}$; (ii) $9^{\frac{3}{2}}$; (iii) $8^{\frac{1}{3}}$; (iv) $27^{\frac{2}{3}}$;
(v) $64^{\frac{4}{3}}$; (vi) $16^{1{\cdot}25}$; (vii) $81^{\frac{3}{4}}$; (viii) $32^{\frac{3}{5}}$.

9. What are the values of :

(i) $8^{-\frac{1}{3}}$; (ii) $4^{-\frac{3}{2}}$; (iii) $16^{-\frac{3}{4}}$; (iv) $100^{-1{\cdot}5}$;
(v) $(\frac{1}{9})^{-\frac{1}{2}}$; (vi) $(2\frac{1}{4})^{-\frac{3}{2}}$; (vii) $1^{-\frac{1}{3}}$; (viii) $\left(\dfrac{27}{1000}\right)^{-\frac{2}{3}}$.

10. Express the following facts without using negative indices :

(i) The wave-length of yellow light is 27×10^{-6} inches.
(ii) The diameter of an electron is 3×10^{-13} mm.
(iii) The time between two collisions of molecules of hydrogen is $1{\cdot}06 \times 10^{-10}$ seconds.

11. Express the following without using fractional or negative indices :

(i) lt^{-2} ; $ml^{-3}t^{-1}$; (ii) $d=kt^{-3}$; $\mathsf{R}=\mu^{-\frac{1}{2}}a^{\frac{1}{2}}$.

(iii) $\dfrac{l}{\mathsf{L}}\left(\dfrac{\mathsf{T}}{t}\right)^{-2}$; $\left(\dfrac{m_1}{m_2}\right)^{-\frac{1}{2}}$; (iv) $d=k\mathsf{H}^{\frac{1}{2}}\mathsf{N}^{-\frac{1}{4}}$.

12. Write the following so that the denominator is 1 :

(i) $\dfrac{6}{a^3}$; (ii) $\dfrac{4}{b}$; (iii) $\dfrac{1}{cd}$; (iv) $\dfrac{5}{x\sqrt{y}}$.

13. Express as powers of x, *i.e.* in the form x^n,

(i) $\sqrt{x^3}$; (ii) $\sqrt[3]{x}$; (iii) $\sqrt{\left(\dfrac{1}{x}\right)}$; (iv) $\sqrt{\left(\dfrac{1}{x^5}\right)}$.

14. Simplify :

(i) $3x^{\frac{1}{3}}\times 3x^{\frac{1}{2}}$; (ii) $y^{\frac{1}{4}}\times y^{\frac{3}{4}}$; (iii) $z\div\sqrt{z}$;
(iv) $\sqrt{a^3}\div\sqrt{a}$; (v) $\sqrt{b}\div\sqrt[3]{b}$; (vi) $c^{-2}\times c^6$.

15. (i) Multiply $a+a^{-1}$ by a ; (ii) Divide b^{-10} by b^{-5}.

16. Simplify (i) $(x^2)^{-2}$; (ii) $(y^{-3})^{-3}$; (iii) $(z^{-6})^{-\frac{1}{3}}$.

EXERCISE F.P. 2

Logarithms, Positive Indices. (Chapter I)

Evaluate correct to 3 significant figures :

1. $2{\cdot}76\times2{\cdot}13$.
2. $8{\cdot}69\div3{\cdot}58$.
3. $48{\cdot}17\div5{\cdot}462$.
4. $17{\cdot}36\times6{\cdot}024$.
5. 409×207.
6. $53{\cdot}07\div3{\cdot}028$.
7. $8607\div60{\cdot}09$.
8. $(30{\cdot}07)^2$.
9. $(7{\cdot}006)^3$.
10. $1000\div3{\cdot}807$.
11. $(10{\cdot}07)^4$.
12. $100\div2{\cdot}904$.
13. $\dfrac{42{\cdot}3\times6{\cdot}76}{8{\cdot}27}$.
14. $\dfrac{782{\cdot}4}{5{\cdot}076\times73{\cdot}04}$.
15. $\dfrac{528\times53{\cdot}9}{70{\cdot}3\times62{\cdot}4}$.
16. $\dfrac{(7{\cdot}327)^2}{10{\cdot}03\times3{\cdot}452}$.
17. $\dfrac{37{\cdot}25\times1000}{(66{\cdot}4)^2\times1{\cdot}01}$.
18. $\dfrac{5{\cdot}17\times86{\cdot}2\times9{\cdot}06}{4{\cdot}68\times2{\cdot}03\times7{\cdot}74}$.
19. $\dfrac{48{\cdot}03\times60{\cdot}07}{(31{\cdot}62)^2}$.
20. $\dfrac{2\frac{1}{4}\times86{\cdot}27\times11\frac{3}{4}}{6\times(6{\cdot}23)^2}$.
21. (i) $\sqrt{6{\cdot}725}$; (ii) $\sqrt{67{\cdot}25}$; (iii) $\sqrt{672{\cdot}5}$.
22. (i) $\sqrt[3]{6{\cdot}725}$; (ii) $\sqrt[3]{67{\cdot}25}$; (iii) $\sqrt[3]{672{\cdot}5}$.

23. $\dfrac{\sqrt{864\cdot7}}{(3\cdot726)^2}$. 24. $\sqrt{\left(\dfrac{37\cdot16}{8\cdot49}\right)}$. 25. $\dfrac{\sqrt{7238}}{\sqrt[3]{8046}}$.

26. $\sqrt{\left\{\dfrac{6\cdot087\times153}{11\cdot04\times1\cdot11}\right\}}$. 27. $\dfrac{86\times\sqrt{7\cdot07}}{(27\cdot96)^{\frac{3}{4}}}$.

EXERCISE F.P. 3

Logarithms, General Indices (Chapter I)

Find, correct to 3 significant figures:

1. $0\cdot527\times8\cdot34$. 2. $0\cdot0846\times6\cdot15$. 3. $0\cdot00472\times0\cdot148$.

4. $0\cdot0836\times0\cdot729$. 5. $6\cdot27\div8\cdot54$. 6. $0\cdot726\div4\cdot37$.

7. $0\cdot0472\div0\cdot624$. 8. $0\cdot372\div0\cdot00828$. 9. $(0\cdot7273)^2$.

10. $(0\cdot04628)^3$. 11. $0\cdot2074\div0\cdot00089$. 12. $1\div0\cdot001008$.

13. (i) $\sqrt{0\cdot8279}$; (ii) $\sqrt{0\cdot08279}$; (iii) $\sqrt{0\cdot008279}$.

14. (i) $\sqrt[3]{0\cdot7468}$; (ii) $\sqrt[3]{0\cdot07468}$; (iii) $\sqrt[3]{0\cdot007468}$.

15. $\dfrac{0\cdot785\times0\cdot472}{18\cdot06}$. 16. $\dfrac{364\cdot7\times0\cdot8275}{7693}$.

17. $\dfrac{0\cdot0627\times5\cdot164}{0\cdot7173\times0\cdot05}$. 18. $\dfrac{1}{(0\cdot8423)^2\times230\cdot6}$.

19. $\sqrt{\left(\dfrac{55\cdot7\times0\cdot75}{9094\times23}\right)}$. 20. $\dfrac{114\times\sqrt{0\cdot727}}{(0\cdot4521)^2\times0\cdot1}$.

21. $\dfrac{\sqrt[3]{0\cdot01}}{(72\cdot14)^2\times3\frac{1}{7}}$. 22. $\dfrac{\sqrt{0\cdot5}\times\sqrt[3]{0\cdot6}\times\sqrt[4]{0\cdot7}}{(0\cdot765)^{\frac{2}{3}}}$

23. $(0\cdot727)^{-3}$. 24. $(0\cdot084)^{1\cdot5}$. 25. $(0\cdot066)^{-0\cdot2}$.

EXERCISE F.P. 4

Surds. (Chapter III)

Simplify the following:

1. $\sqrt{6}\times\sqrt{24}$. 2. $\sqrt{32}\times\sqrt{18}$. 3. $\sqrt{18}\div\sqrt{200}$.

4. $\sqrt{72}-\sqrt{18}$. 5. $\sqrt{5}-\sqrt{0\cdot8}$. 6. $\dfrac{6}{\sqrt{2}}+\dfrac{16}{\sqrt{8}}$.

7. $\sqrt{3}(\sqrt{3}-1)$. 8. $\sqrt{2}(\sqrt{6}-\sqrt{2})$. 9. $\sqrt{3}(\sqrt{27}-\sqrt{3})$.

10. $(\sqrt{2}+1)(\sqrt{2}+2)$. 11. $(2\sqrt{5}-3)(\sqrt{5}+1)$.

12. $(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$. 13. $(2\sqrt{5}-3\sqrt{10})^2$.

14. $(\sqrt{a}-\sqrt{b})^2$. 15. $\left\{\sqrt{\left(\dfrac{c}{d}\right)}+\sqrt{\left(\dfrac{d}{c}\right)}\right\}^2$.

16. $\dfrac{1}{\sqrt{\frac{8}{9}}+\sqrt{2}}$. 17. $\dfrac{12}{\sqrt{5}+\sqrt{2}}$. 18. $\dfrac{2\sqrt{3}}{2\sqrt{3}-3}$.

19. $\dfrac{3}{3\sqrt{2}-2}-\dfrac{2}{3-\sqrt{2}}$. 20. $\dfrac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}+\dfrac{\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

21. Evaluate $\left(x+\dfrac{1}{x}\right)^2$ if $x=\sqrt{7}-\sqrt{6}$.

22. Solve $x\sqrt{2}+x=7$; answer to 3 figures.

23. If x, y are rational and if $x+y\sqrt{3}=(\sqrt{3}-1)^3$, find x and y.

24. Find the square root of (i) $29-12\sqrt{5}$; (ii) $24-12\sqrt{3}$.

Solve the equations :

25. $x=\sqrt{(2x+3)}$. 26. $x-\sqrt{(x+4)}=2$.

27. $\sqrt{(x+1)}=3+\sqrt{(x-2)}$. 28. $\sqrt{(3x+1)}-\sqrt{(2x-1)}=\sqrt{x}$.

29. Simplify $(\sqrt{5}+2)\div\sqrt{(9-4\sqrt{5})}$.

30. Express with a rational denominator, $\dfrac{1}{\sqrt{3}+2\sqrt{2}+\sqrt{5}}$.

31. Solve (i) $\sqrt[3]{(2x-1)}=3$; (ii) $\sqrt[3]{x}=\frac{1}{2}\sqrt{x}$.

32. If $a=\sqrt{\left(\dfrac{e-1}{e+1}\right)}$, express $\dfrac{1-a}{1+a}$ in terms of e as simply as possible.

EXERCISE F.P. 5

Series. (Chapter V)

1. How many terms are there in the following series ?

(i) 22, 29, 36, ..., 162, 169 ; (ii) 100, 97, 94, ..., 43, 40.

2. Find the least number of three digits which belongs to the series (i) 5, 11, 17, ... ; (ii) $\frac{3}{4}$, $1\frac{1}{2}$, 3, 6, ...

3. The 1st term of an A.P. is 10, the 3rd term is 11 ; find the 20th.

4. The 1st and last terms of an A.P. are 21, 193 ; the common difference is 2 ; find the number of terms and their sum.

Sum the following series, Nos. 5-14 :

5. 7, 8, 9, 10, ... (100 terms).

6. 22, 19, 16, 13, ... (20 terms).

7. $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, ... (8 terms).

8. 7, -14, 28, -56, ... (11 terms).

9. 5·3, 7·2, 9·1, 11, ... (25 terms).

10. $8\frac{1}{3}$, $6\frac{5}{6}$, $5\frac{1}{3}$, $3\frac{5}{6}$, ... (18 terms).

11. 27, -12, $5\frac{1}{3}$, $-2\frac{10}{27}$, ... (20 terms).

12. 5, 2, 0·8, 0·32, ... (12 terms).

13. 1, 3, 5, 7, ... (n terms).

14. 2, 4, 6, 8, ... ($2n$ terms).

15. Simplify (i) $8(-\frac{1}{2})^7$; (ii) $(-1)^{2n}-(-1)^{2n-1}$, n integral.

16. Write down, using the index notation, the nth term of a G.P. if the first two terms are

(i) 8, 4; (ii) 9, -3; (iii) $-a$, a^3; (iv) b^3, 1; (v) 144, 96.

17. The 7th term of an A.P. is 5 and the 9th term is $5\frac{2}{3}$; find the 25th term.

18. Find the sum of 30 terms of an A.P. whose 3rd term is 6, and whose 8th term is 9.

19. A man saves £80 his first year of work, and each year afterwards saves £15 more than in the preceding year. How much does he save in 10 years?

20. Find, using logarithms, an approximate value of

(i) $50+50(1{\cdot}05)+50(1{\cdot}05)^2+\ldots$ to 10 terms.

(ii) $8+12+18+27+\ldots$ to 18 terms.

21. A firm sends out a set of sample iron bolts, the difference in weight of successive sizes being the same. The smallest weighs $1\frac{1}{2}$ oz., the largest $10\frac{1}{2}$ oz., and the total weight of a complete set is 4 lb. 14 oz. What is the weight of the largest bolt but one?

22. A man starting business loses £240 the first year, £160 the second year, £80 the third year; if the same improvement continues, what is his total gain or loss after 12 years?

23. Find the sum, $0{\cdot}7+0{\cdot}71+0{\cdot}72+0{\cdot}73+\ldots$ to 100 terms.

24. The first two terms of an H.P. are 10, 9; what is the 10th term?

25. Express $0{\cdot}\dot{4}\dot{5}$ as a G.P. and then find its limiting sum.

26. Insert three harmonic means between 2 and 6.

27. The height of a tree is 20 feet, and one year later is 24 feet; its growth each year is $\frac{3}{4}$ of its growth the previous year. What is the limiting height of the tree, however long it lives?

EXERCISE F.P. 6

Ratio and Proportion. (Chapter VI)

1. By deducting discount at the rate of 1s. in the £, a bill of £x is reduced to £y; find $x:y$.

2. If $\frac{1}{a}+\frac{1}{b}=3\left(\frac{1}{a}-\frac{1}{b}\right)$, find $a:b$.

3. If $4x^2+9y^2=12xy$, find $x:y$.

4. The ratio of the radii of two circles is $p:q$ and the ratio of their areas is $(p-x):(q-x)$; find x in terms of p, q.

5. The ratio of a man's expenditure to his income is

$$(1-n):(1+n);$$

find the ratio of his savings to his expenditure.

6. The incomes of A and B are in the ratio 5 : 4; their expenditures are in the ratio 6 : 5, and their savings are in the ratio 10 : 7. Find the ratio of A's income to B's expenditure.

7. If $\frac{x+y+z}{11}=\frac{x+y-z}{8}=\frac{x-y}{5}$, find $x:y:z$.

8. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{ac}{bd}=\frac{a^2-3c^2}{b^2-3d^2}$.

9. If $\frac{x+y}{11}=\frac{x-y}{5}$, find $\frac{x^2-y^2}{xy}$.

10. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a+c}{b+d}=\sqrt{\left(\frac{a^2+c^2}{b^2+d^2}\right)}$.

11. If b is the mean proportional between a, c, prove that $\left(\frac{a}{b}\right)^2=\frac{a}{c}$.

12. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $\left(\frac{b}{c}\right)^3=\frac{a}{d}$.

13. If $\frac{x^2}{p}=\frac{x}{q}=\frac{1}{r}$, find p in terms of q, r.

14. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, state which of the following relations must be true:

(i) $\frac{a+c}{b+d}=\frac{c-e}{d-f}$; (ii) $\frac{a-c}{c-e}=\frac{b+d}{d+f}$; (iii) $\frac{ac}{bd}=\frac{e}{f}$;

(iv) $\frac{3a+c}{3b+d}=\frac{2c-f}{2d-e}$; (v) $\sqrt{\left(\frac{a^2-e^2}{b^2-f^2}\right)}=\frac{c}{d}$;

(vi) $\frac{a^3-a^2c}{b^3-b^2d}=\frac{c^3+e^3}{d^3+f^3}$; (vii) $\frac{2a^2-7e}{2b^2-7f}=\left(\frac{c}{d}\right)^2$.

15. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $b+c$ is a mean proportional between $a+b$ and $c+d$.

16. If $\frac{a}{b} > \frac{c}{d}$ and if a, b, c, d are positive, prove that

$$\frac{a}{b} > \frac{a+c}{b+d} > \frac{c}{d}.$$

17. If $\frac{a}{x-y} = \frac{b}{y-z} = \frac{c}{x-z}$, prove that $a+b=c$.

18. Solve $\frac{x^2-2x+5}{3x^2+4x-1} = \frac{x^2+2x-5}{3x^2-4x+1}$.

19. If $\frac{x}{y} + \frac{y}{x} = \frac{a}{b}$, express $\left(\frac{x+y}{x-y}\right)^2$ in terms of a, b.

20. If $p^2+q^2=1$ and $\frac{p}{a} = \frac{q}{b}$, express pq in terms of a, b.

EXERCISE F.P. 7

Factors and Fractions (Chapter VIII)

Find the factors of the following :

1. c^3+1.
2. x^6-y^6.
3. xy^3-yx^3.
4. x^3y^3-1.
5. $27a^3-8b^3$.
6. $1+a^3b^6$.
7. $(c+d)^3-1$.
8. $8a^3-a^6$.
9. $(a+b)^3-(a+b)$.
10. x^2-y^2-2x+1.
11. $a^2x^2+2abx+b^2-c^2$.

Simplify the following :

12. $\dfrac{x^2-xy-x+y}{x^4-x^2y^2-x^2+y^2}$.

13. $\dfrac{(a-b)^2-9c^2}{a^2-(b+3c)^2}$.

14. $a - \dfrac{a^2-1}{a^2+\dfrac{1}{a}}$.

15. $\left(a-\dfrac{b^2}{a}\right)\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right) \div \left(\dfrac{a^3}{b^2}-\dfrac{b^2}{a}\right)$.

16. $\dfrac{x^3+y^3}{x^2-xy+y^2} - \dfrac{x^3-y^3}{x^2+xy+y^2}$.

17. $\left(a^2-\dfrac{b^3}{a}\right) \div \left\{\dfrac{(a+b)^2}{ab}-1\right\}$.

18. $\left(x^2+4y^2+\dfrac{15y^4}{x^2-4y^2}\right) \div \left(x+2y+\dfrac{3y^2}{x-2y}\right)$.

19. $(x+y-z)^2-(x-y+z)^2+(x-y-z)^2-(x+y+z)^2$.

20. $\left(\dfrac{x}{y}+\dfrac{y}{x}-2\right) \div \left(\dfrac{1}{x}-\dfrac{1}{y}\right)^2$.

21. $\dfrac{ab}{a^3+b^3} + \dfrac{a+b}{a^2-ab+b^2} - \dfrac{1}{a+b}$.

22. $\dfrac{\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{2}{ab}}{a+b} - \dfrac{\dfrac{1}{a^2}+\dfrac{1}{b^2}-\dfrac{2}{ab}}{a-b}$.

23. $\left(\frac{x+y}{x-y}+\frac{x-y}{x+y}\right)\div\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)$.

24. $\left(\frac{a}{b}-1+\frac{b}{a}\right)\left(\frac{a}{b}+2+\frac{b}{a}\right)\div\left(\frac{a^2}{b}+\frac{b^2}{a}\right)$.

25. $\left(\frac{a}{b}-\frac{a-b}{a+b}\right)\left(\frac{a}{a-b}-1\right)\div\left\{\left(\frac{b}{a}-\frac{b+a}{b-a}\right)\left(\frac{a}{a+b}-1\right)\right\}$.

26. $\frac{a(b+c)}{(a-b)(a-c)}+\frac{b(c+a)}{(b-c)(b-a)}+\frac{c(a+b)}{(c-a)(c-b)}$.

27. Expand $(a-b+c-d)(a-b-c+d)$.

28. Solve $\frac{6}{x-4}+\frac{4}{x-2}=\frac{1}{x-5}+\frac{9}{x-3}$.

29. Solve $(a+b)x+(a-b)y=(a+b)y-(a-b)x=a(a^2+b^2)$.

30. Solve $\left(\frac{x-1}{x+1}\right)^2=1-\frac{4}{x}$.

31. If $c^2=c+1$, prove that $c^5=5c+3$.

32. Factorise $(b-c)(x-a)^2+(c-a)(x-b)^2+(a-b)(x-c)^2$.

33. Prove that $(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a)$.

34. Simplify $\left\{\frac{a-b}{(a+b)(c+b)}+\frac{1}{b+c}\right\}\left\{a+b+c+\frac{ac}{b}\right\}$.

35. If $\frac{b}{b+c}+\frac{c}{c+a}+\frac{a}{a+b}=\frac{c}{b+c}+\frac{a}{c+a}+\frac{b}{a+b}$, prove that

$$(b-c)(c-a)(a-b)=0.$$

EXERCISE F.P. 8

Literal Equations (Chapter IX)

Solve for x the following equations:

1. $ax=b-ax$. 2. $3(x+a)=5(a-x)$. 3. $a+\frac{1}{x}=c$.

4. $bx=1-cx$. 5. $x=b-cx$. 6. $\frac{a}{x}=ab+ac$.

7. $ax-a^2=bx-b^2$. 8. $ax+a=bx+b$. 9. $\frac{x+p}{p}=\frac{x-q}{q}$.

10. $\frac{ax}{b}-1=\frac{bx}{a}+\frac{a}{b}$. 11. $(x-p)^2=(x-q)^2$.

12. $(x-a)(x-b)=(x+a)(x+b)$. 13. $(x-a)^2=c^2$. 14. $\frac{x}{a}=\frac{b}{x}$.

Find x and y from the following simultaneous equations :

15. $x+2y=5a$, $2x-y=2a$.
16. $ax+by=c$, $x-y=1$.
17. $x+y=2b$, $ax-by=a^2+b^2$.

18. $x=py$, $ax+y=c$.
19. $px+qy=q^2$, $qx-py=p^2$.
20. $\dfrac{x}{p}+\dfrac{y}{q}=1$, $px-qy=p^2$.

Solve for x the following equations :

21. $x^2+6a^2=5ax$.
22. $x^2=3bx+28b^2$.
23. $x^2-(c+d)x+cd=0$.
24. $x^2-2px=q^2-p^2$.
25. $x^2+2x=a^2-1$.
26. $ab(x^2+1)=x(a^2+b^2)$.
27. $ax^2+(1+ab)x+b=0$.
28. $ax^2-c^2x=a^2x-ac^2$.
29. $(a+b)x^2+(a+2b)x+b=0$.
30. $(x+a)(x-a^2)+4a^4=2a(x+a^2)$.
31. $\dfrac{2b}{x-a}=1+\dfrac{a}{x-b}$.
32. $\dfrac{ax}{b}-\dfrac{b}{ax}=b-\dfrac{1}{b}$.

The Rectangle contained by two Lines

If AB and CD are two given straight lines, "the rectangle contained by AB and CD" means the area of a rectangle whose

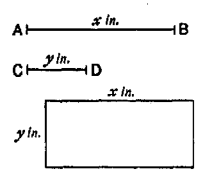


FIG. 11.

length equals AB and whose breadth equals CD, and is written AB . CD. This rectangle is shown in Fig. 11, and we have AB . CD $=xy$ sq. inches.

Similarly AB^2 means the area of a square whose side equals AB, and we have $AB^2=x^2$ sq. inches.

Example. In Fig. 12, O is the mid point of a line AB, and P is any point on AB. Prove that $AP^2+PB^2=2AO^2+2OP^2$.

FIG. 12.

Let AO $=x$ in. and OP $=y$ in.; then also OB $=x$ in.

$\therefore$ AP $=(x+y)$ in. and PB $=$ OB $-$ OP $=(x-y)$ in.

$$\begin{aligned}\therefore \text{AP}^2+\text{PB}^2 &= (x+y)^2+(x-y)^2 \text{ sq. in.}\\ &= x^2+2xy+y^2+x^2-2xy+y^2 \text{ sq. in.}\\ &= 2x^2+2y^2 \text{ sq. in.}\\ &= 2\text{AO}^2+2\text{OP}^2.\end{aligned}$$

EXERCISE F.P. 9

Which of the following relations are equations and which are identities in Nos. 1-6:

1. $(x-1)=-(1-x)$.
2. $(x+1)=-(1+x)$.
3. $x^2-1=(x-1)^2$.
4. $(x-1)^2=(1-x)^2$.
5. $(x+1)^2-(x-1)^2=4x$.
6. $(2x-1)^2=(x-2)^2$.

Under what conditions are the relations in Nos. 7-13 true?

7. $2(x-3)+5(x+1)=3(2x-1)+2(x-2)$.
8. $2(x-3)+5(x+1)=3(5x+4)-2(4x+5)$.
9. $2(x-3)+5(x+1)=3(5x+9)-4(2x+7)$.
10. $x=\dfrac{1}{x}$.
11. $xy-3x-2y+6=0$.
12. $(x+y)^2=x^2+y^2$.
13. $(x-3)^2+(y-4)^2=0$.

Prove the following identities, Nos. 14-20:

14. $(x-y)^2\equiv(y-x)^2$.
15. $(a-b)(x-y)\equiv(b-a)(y-x)$.
16. $x(y-z)+y(z-x)+z(x-y)\equiv 0$.
17. $(a+b)^2-(a-b)^2\equiv 4ab$.
18. $(x+y)(x-y)-z^2\equiv(x+z)(x-z)-y^2$.
19. $(a-x)(b-c)+(b-x)(c-a)+(c-x)(a-b)\equiv 0$.
20. $(a^2+b^2)(c^2+d^2)\equiv(ac+bd)^2+(ad-bc)^2$.

21. ABCD is a straight line such that AB $=$ BC $=$ CD; prove that AD^2 $=$ AB^2 $+$ 2BD^2.

22. With the data of No. 21, prove that AC^2 $-$ CD^2 $=$ AB . AD.

23. A straight line AB is bisected at O and produced to any point P, prove that AP^2 $+$ BP^2 $=$ 2AO^2 $+$ 2OP^2.

24. With the data of No. 23, prove that AP^2 $=$ 4AO . OP $+$ BP^2.

25. If ABCD is a straight line, and if AB $=$ CD, prove that AD^2 $+$ BC^2 $=$ 2AB^2 $+$ 2BD^2.

26. If ABCD is any straight line, prove that

AC . BD $=$ AB . CD $+$ AD . BC.

27. If a line AB is produced to any point P, prove that

$$AB^2 + AP^2 = 2AB \,.\, AP + PB^2.$$

28. If the lengths of the sides of a triangle are (x^2+1), (x^2-1), $2x$ inches, prove that they satisfy the condition that the triangle is right-angled.

What result is obtained if $x=2$?

29. Prove that the area between two concentric circles of radii a, b inches is $\pi(a+b)(a-b)$ sq. inches.

30. If a room is c feet wide, (c^2-c) feet long and $(c-1)$ feet high, verify that the length of a diagonal of the room is (c^2-c+1) feet long.

31. If a, b, c are three consecutive integers, prove that

(i) $ac = b^2 - 1$; (ii) $a^2 - 2b^2 + c^2 = 2$.

32. If a, b, c, d are four consecutive integers, prove that $bc - ad = 2$.

33. If $x = p + \frac{1}{p}$ and $y = p - \frac{1}{p}$, prove that $x^2 - y^2 = 4$.

34. If $2s = a+b+c+d$ and if $a+c = b+d$, prove that

$$(s-a)(s-b)(s-c)(s-d) = abcd.$$

35. P is a point on a line AB such that $AP^2 = AB \,.\, PB$; if $AB = l$ in., $AP = y$ in., find a relation between l and y. Use this to prove that $AB^2 + PB^2 = 3AP^2$.

36. With the data of No. 35, prove that $AP \,.\, PB = AP^2 - PB^2$.

37. O is the mid point of a line AB ; AB is produced to P so that $OB \,.\, OP = BP^2$; prove that $PA^2 = 5PB^2$.

38. In Fig. 13, prove that $AB^2 = BC^2 + AC^2 - 2BC \,.\, CD$. [Let $AB = x$ in., $BC = y$ in., $CA = z$ in., $CD = p$ in., $AD = h$ in., and use Pythagoras' theorem twice.]

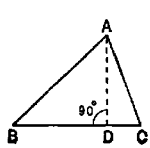


FIG. 13.

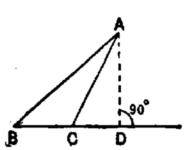


FIG. 14.

39. In Fig. 14, prove that $AB^2 = BC^2 + AC^2 + 2BC \,.\, CD$. [Use the method of No. 38.]

40. If, in Fig. 15, the radii of the circles are a, b inches, prove that $PQ = 2\sqrt{(ab)}$.

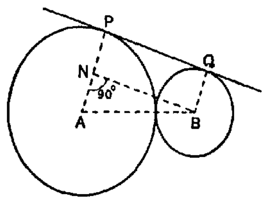


FIG. 15.

EXERCISE F.P. 10

Roots of Quadratic Equations (Chapter IX)

1. If α, β are the roots of $3x^2 + 4x = 2$, find the values of

$$\text{(i) } \frac{1}{\alpha}+\frac{1}{\beta}; \qquad \text{(ii) } (\alpha - 3\beta)(\beta - 3\alpha).$$

2. If α, β are the roots of $2x^2 = 5x + 4$, find the values of

$$\text{(i) } \frac{1}{\alpha^2}+\frac{1}{\beta^2}; \qquad \text{(ii) } (\alpha + 2)(\beta + 2).$$

3. One root of $3x^2 - 8x + c = 0$ is three times the other root. Find c.

4. Find b if $2x^2 + bx + 18 = 0$ has equal roots.

5. Form the equation whose roots are

(i) $1\frac{1}{2}$, $-\frac{2}{3}$; (ii) ± 6; (iii) 0, -1;
(iv) $2c$, $-3c$; (v) $5 - 2\sqrt{3}$, $5 + 2\sqrt{3}$.

6. What is the nature of the roots of

(i) $9x^2 + 25 = 30x$; (ii) $9x^2 + 16 = 30x$;
(iii) $9x^2 + 29 = 30x$?

7 If α, β are the roots of $2x^2 - 3x - 4 = 0$, form the equation whose roots are

$$\text{(i) } \frac{1}{\alpha}, \frac{1}{\beta}; \qquad \text{(ii) } \alpha+\frac{1}{\beta}, \beta+\frac{1}{\alpha}.$$

8. If α, β are the roots of $px^2 + 2qx + r = 0$, express in terms of p, q, r, (i) $\alpha^2 + \beta^2$; (ii) $\alpha^3 + \beta^3$.

9. If $\dfrac{a}{a-x}+\dfrac{b}{b-x} = 2$, show, without solving, that $a > b > x > 0$ is impossible. Then solve the equation.

10. If $a > b > 0$, prove that $x(x-a) = c(x-b)$ has real roots.

11. If α, β are the roots of $\dfrac{1}{x+p} = qx$, find in terms of p, q,

(i) $\alpha\beta$; (ii) $(\alpha-\beta)^2$; (iii) $\dfrac{\alpha}{\beta^2}+\dfrac{\beta}{\alpha^2}$.

12. If α, β are the roots of $(px+q)^2 = prx$, find the condition that
(i) $\alpha = -\beta$; (ii) $\alpha = \frac{2}{3}\beta$; (iii) $\alpha^2+\beta^2 = 3\alpha\beta$.

13. Form the equation whose roots are three-quarters of the roots of $ax^2+bx+c=0$.

14. Find the value of c if the simultaneous equations, $3x+4y=c$, $x^2+y^2=4$ have one, and only one, solution.

15. Prove that $x = c(x-1)(x+2)$ has real roots for all real values of c.

16. Prove that $x = c(x-1)(x-2)$ has real roots, only if the value of c does not lie between $-3-2\sqrt{2}$ and $-3+2\sqrt{2}$.

EXERCISE T. 1

Application of Logarithms. (Chapter I)

[In this exercise, take $\log \pi = 0{\cdot}4971$. Give answers correct to 3 significant figures.]

1. The volume of a sphere, radius r cm., is $\frac{4}{3}\pi r^3$ cu. cm.
   (i) Find the volume of a sphere of radius 1·076 cm.
   (ii) Find the radius of a sphere of volume 4 cu. in.

2. A pendulum of length l feet makes one complete oscillation in $2\pi\sqrt{(l \div 32{\cdot}2)}$ seconds.
   (i) Find the time of oscillation if the length is 50 ft.
   (ii) Find the length if the time of oscillation is 5 seconds.

3. The horse-power H of a steam turbine is given by $\mathrm{H} = \dfrac{6\mathrm{D}^2\mathrm{U}^2}{10^8\mathrm{V}}$; find H, if $\mathrm{D} = 2{\cdot}4$ (mean diameter in ft.), $\mathrm{U} = 1530$ (blade speed in ft./sec.), $\mathrm{V} = 285$ (specific volume of steam in cu. ft.)

4. If $\mathrm{M} = \dfrac{\pi f d^3}{32{\cdot}2}$, find M when $f = 8{\cdot}07$, $d = 2{\cdot}48$.

5. If $d = k\sqrt[3]{\left(\dfrac{\mathrm{H}}{\mathrm{N}}\right)}$, find d when $k = 173$, $\mathrm{H} = 49$, $\mathrm{N} = 3$.

6. If $\mathrm{W} = \dfrac{\mathrm{H}m}{\mathrm{D}^2\mathrm{N}^3}$, find W when $\mathrm{N} = 160$, $\mathrm{D} = 5{\cdot}25$, $\mathrm{H} = 18$, $m = 1{\cdot}43 \times 10^7$.

7. If $\mathrm{V} = \pi l(\mathrm{R}^2 - r^2)$, find V when $l = 0{\cdot}84$, $\mathrm{R} = 0{\cdot}745$, $r = 0{\cdot}23$.

8. Find the value of $\mathrm{A} \,.\, e^{kt}$ when $\mathrm{A}=6{\cdot}35$, $e=2{\cdot}718$, $k=3$, $t=\frac{1}{6}$.

9. If $n=\dfrac{a\mathrm{H}^{\frac{1}{4}}}{p^{\frac{1}{3}}}$, find n when $a=31{\cdot}5$, $\mathrm{H}=10$, $p=160$.

10. The time t seconds for pneumatic transmission through a tube l ft. long, of diameter d ft., under a pressure of p lb. per sq. in., is $t=0{\cdot}000482\sqrt{\left(\dfrac{l^3}{\mathrm{P}d}\right)}$; find t if $l=37{\cdot}5$, $d=0{\cdot}35$, $\mathrm{P}=9{\cdot}2$.

11. From the formula for drilling mild steel, $\mathrm{P}=35{,}500\mathrm{D}^{0\cdot7}\mathrm{T}^{0\cdot6}$, calculate P (thrust in lb.), if $\mathrm{D}=\frac{5}{8}$ (diam. in inches), $\mathrm{T}=\frac{1}{80}$ (feed in inches per rev.).

12. The present value £P of an annuity of £A per annum to last for n years, r per cent. per annum compound interest, is given by $\mathrm{P}=\dfrac{100\mathrm{A}}{r}\left\{1-\left(1+\dfrac{r}{100}\right)^{-n}\right\}$.

(i) Find the present value of an annuity of £250 to last for 10 years, interest 4 per cent.

(ii) What life annuity can a man purchase with £2000 when his expectation of life is 20 years (3 per cent. C.I.) ?

13. If $pv^n=475$, find n when $p=3{\cdot}62$, $v=98{\cdot}5$.

14. If $h=391\mathrm{Q}^n$, find n when $h=7{\cdot}95$, $\mathrm{Q}=0{\cdot}125$.

15. If $3^x=4^y$, find $\dfrac{x}{y}$.

16. Find ϕ from the formula,

$$\phi=\frac{\log t-\log 273}{\log 2{\cdot}718} \text{ when } t=373.$$

EXERCISE T. 2

Variation. (Chapter II)

1. To produce a fixed intensity of illumination, the candle-power c of a lamp must be increased four-fold when the distance x feet from the light is doubled. What is the variation-relation between c and x ?

2. If multiplying x by 10 always produces the effect of dividing y by 100, what is the variation-relation between x and y ?

3. If $y=x+x^2$, in what ratio is y increased when x alters from (i) 1 to 2, (ii) 2 to 4, (iii) 4 to 8 ? Does y vary as any power of x ?

4. If $y \propto \sqrt{x}$, copy and complete the table:

x . .	0	1	2	4	25	100
y . .				4		

5. If $y \propto \frac{1}{\sqrt{x}}$, what is the effect on y of increasing x in the ratio 9 to 4?

6. If $x \propto \frac{1}{y}$ and if $y \propto z^2$, what is the effect on x of doubling z?

7. In the given graph, Fig. 16, y varies as a power of x. What is the formula connecting y and x?

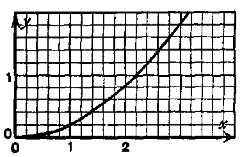


FIG. 16.

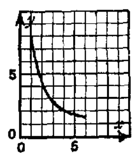


FIG. 17.

8. In the given graph, Fig. 17, y varies inversely as a power of x. What is the formula connecting y and x?

9. The danger distance from the muzzle of a gun, within which the hearing may be injured by the firing, varies as the fifth root of the weight of the charge. For a 5-lb. charge of cordite, the distance is 10 ft.; what is it for 187 lb. of cordite?

10. $S = \frac{6}{x}$ if $y = 2$ and $S = \frac{8}{y^2}$ if $x = 3$; what is the simplest expression for S in terms of x and y?

11. If x varies directly as y^2 and inversely as the square root of z, and if $x = 8$ when $y = 2$ and $z = 9$, find the equation connecting x, y, z.

12. If $A \propto BC^2$ and $B \propto xy^2$ and $C \propto \frac{y}{x}$, what variation-relation connects A with x and y?

13. The horse-power required to propel a ship of given pattern varies jointly as the cube of the speed and as the cube root of the square of its displacement. A ship of 1000 tons displacement is found to require a certain horse-power at 10 knots; by what factor must this horse-power be multiplied in order to propel a similar ship of 8000 tons displacement at 12 knots?

14. The economical diameter for copper wire in an electric circuit varies directly as the square root of the normal current and the fourth root of the cost per horse-power per annum and

inversely as the fourth root of the price of copper per lb. In what ratio should the diameter be altered if the cost of horse-power rises 50 per cent. and the price of copper rises 200 per cent. and if the current is reduced by 25 per cent. ?

15. The velocity, v ft. per sec., of a jet of water varies directly as the weight of water, W lb., delivered per second and inversely as the area of the cross-section, A sq. in., of the jet. Also the horse-power H necessary to produce the jet varies directly as the cube of W and inversely as the square of A. Find (i) how v varies with H and W ; (ii) how v varies with H and A.

16. In a certain machine the effort P lb. required to raise a load W lb. is partly constant and partly varies as the load. For loads of 40 lb., 56 lb., the necessary efforts are 11 lb., 13 lb. ; find the general formula.

17. On a certain building estate near London, the ground rents for plots of land vary partly as the frontage and partly as the frontage and depth jointly.

The ground rent for a plot with 30 ft. frontage and 100 ft. depth is £38, and for a plot with 36 ft. frontage and 140 ft. depth is £60 ; find the general formula. The smallest plot on the estate has a frontage of 24 ft. and a depth of 90 ft. ; what is its ground rent ?

18. The sum of the first n whole numbers, 1, 2, 3, 4, ... varies partly as n and partly as n^2. What is the sum for $n = 2$ and for $n = 3$? Hence find the general formula.

EXERCISE T. 3

Logarithmic Theory and Notation. (Chapter IV)

Simplify the following :

1. $\log 27 - \log 3$. 2. $\log 27 \div \log 3$. 3. $\log 3 - \log \sqrt{3}$.

4. $\log 5 \div \log \sqrt{5}$. 5. $\log 1 \times \log 2$. 6. $\frac{1}{4} \log 16$.

7. $\dfrac{\log (x^2)}{\log x}$. 8. $\dfrac{\log \sqrt{x}}{\log x^2}$. 9. $\dfrac{\log (a^2 b) - \log b}{\log a^3 + \log a}$.

10. Find x if $\dfrac{\log x}{\log 2} = \dfrac{\log 4}{\log 16}$.

11. Express the following in a form which does not involve the logarithmic notation, the logarithmic base being 10 :

(i) $\log x + \log y = 3$; (ii) $\log x + 2 \log y = 3$;
(iii) $x \log 2 = -1$; (iv) $3 \log x - 2 \log y = 1$;
(v) $x \log 5 = \log 6$; (vi) $x \log 5 = y \log 6 + 2$.

12. If $\log 6 = b$ and $\log 12 = c$, express $\log 2$, $\log 3$, $\log 4$ in terms of b, c.

13. Simplify

(i) $\log(1-x^2)-\log(1-x)$; (ii) $\log(1+2x+x^2)\div\log(1+x)$.

14. If $\log(x+y)=\log x+\log y$, find y in terms of x, and prove that $\log x-\log y=\log(x-1)$.

15. Find, without using logarithmic tables, the characteristics of $\log_3 15$, $\log_2 30$, $\log_2 0{\cdot}2$, $\log_9 0{\cdot}001$.

16. Find x if (i) $\log_4 x=1{\cdot}5$; (ii) $\log_4 x=-2{\cdot}5$.

17. Simplify (i) $\log_a b\times\log_b c$; (ii) $a^{\log_a b}$.

18. Simplify $\dfrac{\log_2 5}{\log_2 7}-\dfrac{\log_3 5}{\log_3 7}$.

19. Solve $\log_3 x+\log_3 y=6$, $\log_2 x-\log_2 y=4$.

20. If $5\log_{10} y-2\log_{10} x=4{\cdot}1$, express y in terms of x.

21. Use tables to find the value of $10^{-\frac{1}{16}x^2}$ for x equal to 0, ±1, ±2, ±3, ±4. Draw the graph of the function, and find for what values of x it is equal to half its maximum value.

22. Prove that $\log_a b\times\log_b c\times\log_c a=1$.

EXERCISE T. 4

Harder Indices. (Chapter IV)

Find n from the following equations:

1. (i) $81^n=3$; (ii) $81^n=1$; (iii) $81^n=\sqrt{3}$.

2. (i) $5^n=\frac{1}{25}$; (ii) $5^n=0{\cdot}2$; (iii) $5^n=\dfrac{1}{5\sqrt{5}}$.

3. (i) $(0{\cdot}01)^n=10$; (ii) $(0{\cdot}01)^n=\sqrt{10}$; (iii) $(0{\cdot}01)^n=\sqrt{(0{\cdot}1)}$.

4. (i) $9^n=9.3^{-n}$; (ii) $16^{3+n}\,.\,2^{1+n}=(\frac{1}{2})^{1-n^2}$.

Simplify the following:

5. $1000^{2n}\div100^n$. 6. $\sqrt[4]{b^3}\div(\sqrt[4]{b})^3$. 7. $(y^{-4})^{-\frac{1}{2}}$.

8. $\dfrac{(4x)^{-\frac{1}{2}}}{(2x)^{-2}}$. 9. $\sqrt{y^5}\div y^{-1{\cdot}5}$. 10. $\left(\dfrac{p^{-1}}{3q^2}\right)^{-2}$.

11. $\dfrac{6ab^2}{\sqrt[3]{(8a^{-3})}}$. 12. $\dfrac{cd^{-1}}{\sqrt{(c^{-3}d^{-4})}}$. 13. $\dfrac{4a^{\frac{1}{2}}b^{-2}}{(4a)^{-\frac{1}{2}}b}$.

14. $\dfrac{12^{3n}\,.\,16^{2n}}{48^n}$. 15. $\dfrac{20^{\frac{1}{2}n}\,.\,100^n}{(\sqrt{5})^{3n}}$. 16. $\dfrac{4^n+8^{2n}}{2^n}$.

17. $\sqrt[6]{(ab^4c^3)}\div(a^{\frac{1}{3}}b^{-\frac{1}{3}}c^{-1})$. 18. $(27x^{\frac{3}{2}}y^{-\frac{1}{2}})^{\frac{2}{3}}\div(x^{\frac{1}{2}}y^{\frac{1}{3}})$.

19. $\left(\dfrac{25x^4}{2\cdot25y^6}\right)^{-\frac{1}{2}}$. 20. $\dfrac{9^{2n-1}\,.\,27^{2-n}}{(\sqrt{3})^{6-4n}}$.

Express z in the form cx^py^q, giving c correct to 3 significant figures, in the following cases :

21. $z^3 = \frac{2}{5}xy^2$. 22. $z^4 = \dfrac{3y^2}{7x}$. 23. $9x^2yz = 4$.

24. $x^3 = 6yz^2$. 25. $x = 3\cdot4y^{1\cdot2}z^{-0\cdot6}$. 26. $x^2 = \dfrac{z^{\frac{3}{4}}}{52y^{\frac{1}{4}}}$.

27. Make l the subject of the formula, $r = 0\cdot0315n^{0\cdot4}l^{0\cdot2}p^{-0\cdot25}$, giving the constant factor correct to 2 significant figures.

28. Simplify (i) $(x^{\frac{3}{4}} - x^{-\frac{1}{4}})\sqrt{x}$; (ii) $x^{\frac{1}{6}}(x^{\frac{1}{3}} + x^{-\frac{2}{3}})$.

29. Divide $3x^{\frac{4}{3}} - 6x^{\frac{2}{3}} + 3$ by $x^{\frac{2}{3}}$.

30. Multiply $x + 1 + x^{-1}$ by $x - 1 + x^{-1}$.

31. Multiply $a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1$ by $a^{\frac{1}{3}} - 1$.

32. Divide $b - 1$ by $b^{\frac{1}{2}} - 1$.

33. Divide $c + 1$ by $c^{\frac{1}{3}} + 1$.

34. Simplify $\dfrac{1}{(a+b)^{-2}} \div (a^{-2} - b^{-2})$.

35. If $a^b = b^a$, express ab in the form a^p.

36. If $a^x = b^y = c^z$ and $xy + xz = yz$, find a relation between a, b, c.

37. If x is small compared with 1, $(1+x)^n \simeq 1 + nx$; use this fact to find approximate values of

$$\text{(i) } (1+0\cdot05)^2;\ \text{(ii) } \sqrt{(1\cdot06)};\ \text{(iii) } \frac{1}{1\cdot03};$$

$$\text{(iv) } \frac{1}{\sqrt{(1\cdot008)}};\ \text{(v) } \sqrt[3]{1003}.$$

38. The slope of the graph of $y = x^n$ at $x = c$ is nc^{n-1}. Use this fact to find the slope at $x = c$ of each of the following graphs :

$$\text{(i) } y = x^3;\ \text{(ii) } y = \frac{1}{x^2};\ \text{(iii) } y = \sqrt{x};\ \text{(iv) } y = \frac{1}{\sqrt{x}}.$$

39. The area bounded by the curve $y = x^n$, the x-axis, and the ordinates $x = 0$ and $x = c$, is $\dfrac{c^{n+1}}{n+1}$ units, if $n > -1$. Show that this statement is true if $n = 1$ and if $n = 0$; and use it to find the area for each of the following curves :

$$\text{(i) } y = x^2;\ \text{(ii) } y = \sqrt{x};\ \text{(iii) } y = x\sqrt{x};\ \text{(iv) } y = \frac{1}{\sqrt{x}}.$$

EXERCISE T. 5

Series. (Chapter V)

1. Divide the sum of the odd numbers between 50 and 100 by the sum of the odd numbers less than 50.

2. A Chinese nest of boxes is made so that each box contains all those smaller than it; the outermost weighs 2 oz. and the innermost $\frac{1}{8}$ oz., and their weights decrease by equal amounts. The whole nest weighs 17 oz.; how many boxes are there in it ?

3. Find the sum of all numbers between 100 and 200 which are not divisible either by 2 or by 5.

4. The first term of an A.P. is $5x - 3y$ and the fifth term is $5y - 3x$; find the three intermediate terms.

5. The first, second and nth terms of a G.P. are a, b, l respectively; prove that the sum of n terms is $\dfrac{a^2 - bl}{a - b}$.

6. If the sum of n terms of a series is $5n^2 + 2n$ for all values of n, find the first 3 terms. Find also the rth term and the $(r - 1)$th term, and show that the series is in A.P.

7. A man borrows £1000 and discharges the debt by paying 10 equal annual instalments, the first being paid 1 year after the loan is made. Find to the nearest £ the amount of each instalment, reckoning 4 per cent. per annum compound interest.

8. Prove that $(1 + 2 + 3 + \ldots + n)^2 - (1 + 2 + 3 + \ldots + \overline{n - 1})^2 = n^3$.

9. In an A.P. the 8th term is twice the 4th term, prove that the 9th term is 3 times the 3rd term.

10. The sum of 10 terms of an A.P. is 145 and the sum of its 4th and 9th terms is 5 times its 3rd term. Find the series.

11. What is the first number in the nth bracket of the series,

$$[1],\ [2 + 3],\ [4 + 5 + 6],\ [7 + 8 + 9 + 10],\ \ldots\ ?$$

What is the sum of the numbers inside the nth bracket, and what is the sum of the first n brackets ?

12. Prove that the sum to n terms of $\log a + \log ax + \log ax^2 + \ldots$ is $n \log a + \frac{1}{2}n(n - 1) \log x$.

13. The first and last terms of an A.P. are a, l; there are n terms in all. What is the second term ?

14. If a, b are positive integers and if $a < b$, find the sum of all integers from a to b inclusive.

15. In an A.P., the pth term is $\dfrac{1}{q}$ and the qth term is $\dfrac{1}{p}$; find the (pq)th term and the sum of pq terms.

16. Find the present value of an annuity of £25 to commence in 8 years' time, if there are 15 annual payments in all, reckoning $3\frac{1}{2}$ per cent. per annum compound interest.

17. If $s = 1 + 2x + 3x^2 + 4x^3 + \dots + nx^{n-1}$, express $s - xs$ as a function of x, and so find a simple expression for s in terms of x.

18. Sum to n terms: (i) $9 + 99 + 999 + 9999 + \dots$;
(ii) $2 + 22 + 222 + 2222 + \dots$.

19. In an A.P. of n terms, the sum of the first 2 terms is b and the sum of the last two terms is c, find the sum of n terms.

20. Find the value of the nth bracket and the sum of the first n brackets, of the series :

$$[1],\ [2+4],\ [8+16+32],\ \dots .$$

21. The nth term of a series is $\dfrac{n}{n+1}$; find the sum of the first 3 terms.

22. The first and last terms of a G.P. are a, l; there are n terms in all. Find their product.

23. Find the sum of 198 terms of the series

$$\log 1\tfrac{1}{2} + \log 1\tfrac{1}{3} + \log 1\tfrac{1}{4} + \log 1\tfrac{1}{5} + \dots .$$

24. Prove that $\dfrac{1}{r(r+1)} = \dfrac{1}{r} - \dfrac{1}{r+1}$; use this fact to find the sum of n terms of the series, $\dfrac{1}{1\times 2}, \dfrac{1}{2\times 3}, \dfrac{1}{3\times 4}, \dfrac{1}{4\times 5}, \dots$. Then find the limiting sum of this series.

25. Use the method of No. 24 to find the sum of n terms of the series $\dfrac{1}{3\times 5}, \dfrac{1}{5\times 7}, \dfrac{1}{7\times 9}, \dots$.

26. If all rational numbers are written down as follows,

$$\tfrac{1}{1}, \tfrac{2}{1}, \tfrac{1}{2}, \tfrac{3}{1}, \tfrac{2}{2}, \tfrac{1}{3}, \tfrac{4}{1}, \tfrac{3}{2}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{5}{1}, \tfrac{4}{2}, \tfrac{3}{3}, \tfrac{2}{4}, \tfrac{1}{5}, \tfrac{6}{1}, \dots ,$$

(i) how many terms are there before $\frac{11}{13}$?

(ii) what will be the 100th term?

27. Show that the contents of each bracket in the series

$$[\tfrac{1}{3} + \tfrac{1}{4}],\ [\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}],\ [\tfrac{1}{9} + \tfrac{1}{10} + \dots + \tfrac{1}{16}],\ \dots ,$$

is greater than $\frac{1}{2}$; and then prove that the series,

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \dots , \frac{1}{n}, \dots ,$$

has no limiting sum.

28. Prove that the sum of any number of terms of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$ is not less than $\frac{1}{2}$ and not more than 1.

29. Sum to 20 terms the series $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\frac{9}{16}+\dots$.

30. If a^2, b^2, c^2 are in A.P., prove that $b+c$, $c+a$, $a+b$ are in H.P.

EXERCISE T. 6

Use of Graphs. (Chapter VII)

1. *Sketch* the graphs of $y=\frac{2}{x}$, $y=x^2+3$, $y=x^2-6$, and use them to find the number of roots (not their values) of the equations (i) $x^3+3x-2=0$; (ii) $x^3-6x-2=0$.

Find by inspection the roots of $x^3-3x-2=0$; what can you now say about the graphs of $y=\frac{2}{x}$ and $y=x^2-3$?

2. Draw the graph of $y=\frac{1}{10}(x^3-4x)$ for values of x from -4 to $+4$, and read off answers to the following:

(i) For what positive value of x is x^3-4x a minimum?

(ii) What is the maximum value of x^3-4x if x is negative?

(iii) For what values of x is $\frac{x^3-4x}{10}$ equal to $0{\cdot}2$?

(iv) Solve the equations (*a*) $x^3-4x=1$; (*b*) $x^3-4x+1=0$.

3. Draw the graphs of $y=\frac{x^2}{x+3}$ and $y=1-\frac{x}{2}$ for values of x from -2 to $+6$. What is the quadratic equation whose roots are connected with the intersections of the two graphs? Obtain its roots graphically and by calculation.

4. Draw the graph of $y=\frac{1}{4}x(x+1)(x-4)$ for values of x from -3 to $+5$. Use it to answer the following:

(i) For what values of x does x^3-3x^2-4x equal 6?

(ii) For what range of values of x is x^3-3x^2-4x greater than 6?

(iii) What is the least value of x^3-3x^2-4x if x is positive?

(iv) What is the largest value of x^3-3x^2-4x if x is negative?

(v) Solve the equations:

(*a*) $x^3-3x^2-4x=13$; (*b*) $x^3-3x^2-4x=-4$;

(*c*) $x^3-3x^2-4x+19=0$; (*d*) $x^3-3x^2-4x=16$;

(*e*) $x^3-3x^2-4x+12=0$.

5. Draw as much as your paper allows of the graph of

$$y=\frac{120}{x-30}+20,$$

taking the origin in the middle of the paper, and 1 inch on each axis to represent 20 units.

Solve graphically the simultaneous equations,

$$y=\frac{120}{x-30}+20,\ x-2y+40=0.$$

Check by calculation.

6. Draw the graph of $y=10^x$ for values of x from 0 to 1. By drawing other suitable graphs, solve the equations :

(i) $10^x=10x$; (ii) $10^x=7x$;

(iii) $10^x+10x=6$; (iv) $10^x=10x-1$.

7. Draw the graph of $y=\log x$ for values of x from 1 to 10. By drawing other suitable graphs, solve the equations :

(i) $\log x=\frac{x}{10}$; (ii) $\log x=0\cdot6-0\cdot06x$; (iii) $5\log x+x=4$.

8. (i) What is the value of $\frac{x-1}{x-2}$ if $x=2\cdot01$, if $x=2\cdot001$? What is the value if $x=1\cdot99$, if $x=1\cdot999$?

(ii) What is the value of x if $\frac{x-1}{x-2}=1\cdot01$, if $\frac{x-1}{x-2}=1\cdot001$?

What is the value if $\frac{x-1}{x-2}=0\cdot99$, if $\frac{x-1}{x-2}=0\cdot999$?

(iii) Sketch the graph of $y=\frac{x-1}{x-2}$.

9. (i) For what values of x is $(x-1)(x-3)(x-6)$ zero ?

(ii) For what range of values of x is this function positive ?

(iii) Find the values of the function when $x=11,\ 0,\ -9$.

(iv) Sketch the graph of $y=(x-1)(x-3)(x-6)$.

10. Make a table of values from $x=-2$ to $x=10$ for the function $y=\frac{(x-1)(x-6)}{x-3}$, and show where there are missing values, if any ; then sketch the graph of the function.

11. Make a table of values from $x=-2$ to $x=10$ for the function $y=\frac{x(x-6)}{(x-2)(x-4)}$, and show where there are missing values. Add to the table the values of y when $x=\pm 100,\ \pm 1000$. Then sketch the graph of the function.

12. The base of an open cistern is a square of side x feet ; its volume is 200 cu. ft. If the total area of the base and sides is A sq. ft., prove that $A=x^2+\frac{800}{x}$. Represent the relation between A and x by a graph for values of x from 4 to 12. For what value of x is A least ? For what positive values of x is A equal to 175 ?

13. A sheet of tin is 24 inches square; equal squares, side x in., are cut at the four corners, and the sides are then turned up to make a rectangular box. If the volume of the box is V cu. in., express V in terms of x. Show by a graph the relation between V and x for values of x from 0 to 12. (i) For what value of x is V greatest? (ii) For what positive values of x is V equal to 500?

14. The following observations show the connection between the pressure and volume of saturated steam:

Volume, cu. ft. per lb. water	v	297	173	82·4	55·1	21·3
Pressure, lb. per sq. in. .	p	1·13	2·02	4·42	6·77	18·7

Plot $\log p$ against $\log v$, and then express p in terms of v.

15. Draw the graphs of $y = 1 - x^2$ and $x = \frac{1}{2}(3 - y^2)$ from $x = -2$ to $x = +2$. Thence find approximate solutions of these simultaneous equations. What equation in x can be solved from these graphs? What equation in y can be solved?

16. The graph of $y = a + bx + cx^2$ passes through the origin and through the two points $(1, 1)$; $(2, -2)$. Find the values of a, b, c, and calculate the values of x when $y = -5$. Sketch the graph.

EXERCISE T. 7

Remainder Theorem. (Chapter VIII)

1. Prove that $x + 2y$ is a factor of $x^4 + 10xy^3 + 4y^4$.

2. Find a if $x - 2$ is a factor of $x^3 + ax - 4$.

3. Find a, b if $x^2 + x - 6$ is a factor of $x^3 - ax^2 - bx - 6$.

4. What is the remainder when (i) $x^3 - 1$ is divided by $x + 1$, (ii) $x^4 - x - 2$ is divided by $x - 3$?

5. Factorise (i) $x^3 - 8x^2 + 19x - 12$; (ii) $2x^3 - 3x^2 - 3x + 2$.

6. Solve the equation, $2x^3 = x^2 + 13x + 6$.

7. If n is a positive integer, find the condition that

(i) $x + 1$ is a factor of $x^n + 1$;

(ii) $x + 1$ is a factor of $x^n - 1$.

8. Prove that $x - y$ is a factor of $(x - y)^3 + (y - z)^3 + (z - x)^3$; then factorise the expression.

9. Prove that $x+y$ is a factor of

$$(xy-yz-zx)(x+y-z)+xyz;$$

then factorise the expression.

10. Prove that x and $y+z$ are each factors of

$$(xy-yz+zx)^3-x^3y^3+y^3z^3-z^3x^3;$$

then factorise the expression.

11. Factorise $(x+y+z)^3-(x^3+y^3+z^3)$.

12. Prove that $x+a+b+c$ is a factor of

$$(x+b+c)(x+c+a)(x+a+b)+abc.$$

13. Prove that $x-y$, $y-z$, $z-x$ are each factors of

$$x(y-z)^3+y(z-x)^3+z(x-y)^3;$$

then find the remaining factor.

14. Prove that a is a factor of

$$(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3;$$

then factorise the expression.

15. Factorise

$$(x^2-y^2)(x-y)^2+(y^2-z^2)(y-z)^2+(z^2-x^2)(z-x)^2.$$

EXERCISE T. 8

Literal Relations. (Chapter IX)

1. Make n the subject of the electrical formula, $C=\dfrac{nE}{R+nr}$.

2. If a, b are consecutive even integers, prove that

$$a^2+b^2=2ab+4.$$

3. If $x+y=t+\dfrac{1}{t}$ and $x-y=t-\dfrac{1}{t}$, prove that $xy=1$.

4. Simplify $\left(a+\dfrac{1}{a}\right)^2-\left(a-\dfrac{1}{a}\right)^2$.

5. If $\dfrac{1}{x}+\dfrac{1}{a}=\dfrac{1}{p}$ and $\dfrac{1}{y}-\dfrac{1}{a}=\dfrac{1}{q}$, express $\dfrac{xy}{x+y}$ in a form not containing x or y.

6. If $x+3y+7z=14$ and $x+4y+10z=17$, show that there is a *numerical* value of c for which it is possible to find the *numerical* value of $x+5y+cz$ from the data. What is this value of c?

7. Is it possible to find three consecutive integers a, b, c such that $a^2+c^2=2b^2$?

8. If $a(x+y)=a^2-xy$, express $x+y$ in terms of a, x.

Solve, for x, the following equations :

9. $a(x-b)+b(x-c)+c(x-a)=0$.

10. $a\left(x+\frac{1}{b}\right)=b\left(x+\frac{1}{a}\right)$.

11. $(x+a)^2+(x+b)^2=2x^2$.

12. $(x-a)(x-c)=(x-b)^2$.

13. $c\left(\frac{1}{x}+\frac{1}{d}\right)=d\left(\frac{1}{x}+\frac{1}{c}\right)$.

14. $\frac{a}{x(x+a)}=\frac{1}{x}+\frac{1}{a}$.

Find x, y from the following simultaneous equations :

15. $x-uy+u^2=0$, $x-vy+v^2=0$.

16. $mx-y=b$, $x+my=mb$.

17. If $a+b+c=2s$, prove that $a(2s-a)-b(2s-b)=(a-b)c$.

18. If $x+ay=b$ and $x+by=a$, prove that $x-cy=a+b+c$.

19. Given that $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$, prove that $(u-f)(v-f)=f^2$.

20. If b cows and c sheep cost as much as p cows or q sheep, prove that $\frac{b}{p}+\frac{c}{q}=1$.

21. If $a+b=c$, prove that $a^2+b^2+c^2=2(bc+ca-ab)$.

22. If $\frac{x+a}{x+b}=\frac{b^3}{a^3}$ and $\frac{y+a}{y+b}=\frac{a^4}{b^4}$, express xy in terms of a, b.

23. The result of increasing b by x per cent. is the same as decreasing c by x per cent. Find x in terms of b, c.

24. A, B, C, D are 4 points in order on a straight line. If B is the mid point of AD, and if AB $=k$. CD and AB $=l$. BC, find l in terms of k.

25. From £x ys. zd., where $12>x>z$, subtract £z ys. xd. If the result is £p qs. rd., find the sum of £p qs. rd. and £r qs. pd.

26. If $x=\frac{2t}{1+t^2}$ and $y=\frac{1-t^2}{1+t^2}$, prove that $x^2+y^2=1$.

27. If $x+y+z=0$, prove that $x^2-y^2-z^2=2yz$.

28. If $x+y+z=0$, prove that

$$(x-y-z)(y-z-x)(z-x-y)=8xyz.$$

29. Given that $Pt = m(v-u)$ and $Ps = \frac{1}{2}m(v^2-u^2)$, prove that $s = \frac{1}{2}(u+v)t$.

30. Given that $v = u + at$ and $s = \frac{1}{2}(u+v)t$, prove that

(i) $s = ut + \frac{1}{2}at^2$; (ii) $s = vt - \frac{1}{2}at^2$; (iii) $v^2 = u^2 + 2as$.

31. A circular cylinder rests on a table between two rectangular blocks as shown in Fig. 18; if the diameter of the cylinder equals AC, prove that $AB + BC = 1\frac{1}{4}AC$.

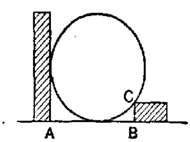


FIG. 18.

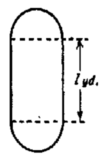


FIG. 19.

32. Fig. 19 represents the inner ring of a race-course, formed of a rectangle with semicircular ends, radii r yd. The perimeter of the ring is C yd. and the area enclosed is A sq. yd. Express A and C in terms of l, r. Prove that $l^2 = \frac{1}{4}C^2 - \pi A$.

33. Fig. 20 is formed by four semicircular arcs; prove that its area is $\frac{1}{4}\pi$. AC . BD. [Take $AB = 2x$ in., $BC = 2y$ in., $CD = 2z$ in.]

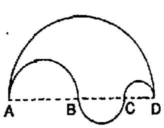


FIG. 20.

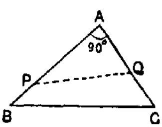


FIG. 21.

34. In Fig. 21, PQ bisects $\triangle ABC$. If $AB = 5$ in., $AC = 4$ in., $AP = x$ in., $PQ = y$ in., express y in terms of x.

35. If $a+b+c=t$, prove that $(at+bc)(bt+ca)(ct+ab)$ is a perfect square.

36. If $\frac{1}{x+y} + \frac{1}{z} = \frac{1}{x} + \frac{1}{y+z}$, prove that either $x = z$ or $y = 0$ or $x+y+z=0$.

37. If $a + \sqrt{a} = 1$, prove that (i) $a + \frac{1}{\sqrt{a}} = 2$; (ii) $a + \frac{1}{a} = 3$.

38. A common type of error when using an adding machine is to record sums such as £7. 5s. as 7s. 5d. or £13. 9s. as 13s. 9d. If only one amount is recorded incorrectly in this manner, and if the error is £x. ys. zd., prove that $x+y+z$ is either 11 or 30.

SUPPLEMENTARY TEST PAPERS Y. 1-10. (Parts I, II.)

Y. 1

1. Simplify (i) $(4a)^2 - 4a^2$; (ii) $\left(\frac{b}{2}\right)^3 - \frac{1}{2}(-b)^3$; (iii) $c + \frac{d}{2} - \frac{c+d}{2}$.

2. Make n the subject of the formula, $\mathrm{P} = \frac{\mathrm{Q} + n\mathrm{R}}{1 - n}$.

3. Factorise (i) $5x^2 - 20$; (ii) $3a^2 + 7ab - 6b^2$.

4. Solve (i) $(x+1)(x-2) - \frac{1}{2}(x-1)(x+2) = \frac{1}{2}(x+3)(x-1)$;
(ii) $x^2 - 12x = 9$ (to 2 places of decimals).

5. A sum of 1 guinea can be made up of either c shillings and d florins or of c sixpences and d half-crowns. Find the values of c and d.

Y. 2

1. (i) Write down the L.C.M. of $2xy(x-y)^2$, $3y^2(x^2-y^2)$, $4x^2(x+y)^2$.
(ii) Find the H.C.F. of $a^3 - 2a^2 - 3a$ and $a^4 - 5a^3 - 6a^2$.

2. Simplify $\left(\frac{1}{a} - \frac{1}{a+b}\right) \div \left(\frac{1}{a} + \frac{1}{b-a}\right)$.

3. Solve (i) $4\mathrm{P} = 5\mathrm{Q}$; $14\mathrm{P} - 13\mathrm{Q} = 1{\cdot}8$; (ii) $2x + \frac{2}{x} = 5$.

4. Find the value of c if $x = \frac{2}{3}$ is a root of $9x^2 - 5cx = 14$. What is the other root ?

5. Two regular polygons are such that the number of sides in one is double that in the other, and an angle of the first is $1\frac{1}{2}$ times that of the second. Find the number of sides of each.

Y. 3

1. (i) What number must be added to $a^2 - 9a$ to make the result a perfect square ?
(ii) Express $(3x-5)(x+11)$ as the difference of two squares.

2. Make a the subject of the formula, $e = \frac{\mathrm{W}(a-b)}{2\mathrm{P}a}$.

3. Simplify (i) $\frac{c}{c+d} - \frac{c+d}{c+2d}$; (ii) $\frac{x^2 - 4x + 3}{x^2 - 5x + 6} \div \frac{x^2 - 1}{x^2 - 4}$.

4. Solve (i) $\frac{1}{x-1} - \frac{2}{x} = \frac{1}{2-x}$; (ii) $x = \frac{1}{y} = \frac{2}{x-1}$.

5. A spring 4 in. long is lengthened $\frac{1}{4}$W in. when a weight W lb. is hung on it. Another spring $3\frac{1}{2}$ in. long is lengthened $\frac{1}{3}$W in. when a weight W lb. is hung on it. For what value of W will the two springs have the same length •

Y. 4

1. $x=\sqrt{(yz^3)}$; $y=2t^4$; $z=2s^2$. Express x in terms of s, t.

2. A man walks S miles up a hill at 3 m.p.h. and the same distance back at 6 m.p.h. What is his average speed ?

3. Factorise (i) $x^2-7x-30$; (ii) $ab-a-b+1$;
(iii) c^2-c^6 ; (iv) $6y^2-23yz-18z^2$.

4. (i) For what values of x is $(x+1)^3$ equal to x^3+1 ?
(ii) Solve $4x-5y-3=9x+2y-20=3y-2x+1$.

5. Eggs are sold either at $3x$ shillings a dozen or at x eggs for a shilling. What is x if these two prices are the same ?

Y. 5

1. Given that $2x-1$ is a factor of $8x^3-1$, find the other factor. What are the factors of $8y^3+1$?

2. Simplify (i) $\{\frac{1}{2}n(n+1)\}^2-\{\frac{1}{2}n(n-1)\}^2$.
(ii) $\dfrac{x^2+11x+30}{x^2-5x-66}\div\dfrac{25-x^2}{x^2-121}$.

3. Make V the subject of the formula, $\dfrac{1}{R}=\dfrac{CL}{t}\cdot\dfrac{V}{V-V_1}$.

4. (i) Solve $Px=88$; $P(x+2)=120$.
(ii) If $4x^2+4x-3=0$ and if $t=2x+1$, find t.

5. The weight of a foot of iron piping, external diameter D in., internal diameter d in., is $2{\cdot}45(D^2-d^2)$ lb. Express this in terms of the mean radius r in., and the thickness t in.

Y. 6

1. Factorise (i) $2a-a^2-1$; (ii) $(xy-1)^2+(x+y)^2$.

2. Find the values of the constants a, b in the formula $\dfrac{1}{w}=\dfrac{a}{u}+\dfrac{b}{v}$, if $w=5$ when $u=4$, $v=10$, and if $w=6$ when $u=3$, $v=9$.

3. (i) Simplify $\dfrac{2}{x^2-1}+\dfrac{3}{x-1}+\dfrac{5}{x+1}$.
(ii) Solve $\dfrac{2}{x^2-1}+\dfrac{3}{x-1}+\dfrac{5}{x+1}=0$.

4. (i) Find x if the squares of $5x+3$ and $5x+4$ are equal.
(ii) Solve $y=x+2$; $x^2+2y=7$.

5. The outer dimensions of a picture frame are 16 in. by 12 in. The width of the frame is the same all round, and the surface area of the frame is 75 sq. in. Find the width.

Y. 7

1. Find the value of $\dfrac{x^{n+1}}{y^{n+1}} - \dfrac{x^n}{y^n}$ when $x=3$, $y=2$, $n=2$.

2. Factorise (i) $36x^2+42x-18$;

   (ii) $a^2c^2-b^2d^2+a^2d^2-b^2c^2$.

3. (i) Make W the subject of the formula, $t=\dfrac{2a\mathrm{W}}{\mathrm{W}+w}\left(1+\dfrac{\mathrm{V}}{v}\right)$

   (ii) If a and b are two positive numbers such that

   $$a^2+ab+b^2=79 \text{ and } a^2+b^2=58,$$

   calculate in succession ab, $a^2+2ab+b^2$, $a+b$, $a-b$.

4. (i) Solve $z=\dfrac{1}{z+1}$ (to 2 places of decimals).

   (ii) $y=2x-1$; $y^2=8x+1$.

5. $\frac{1}{2}(x-1)$ and $\frac{1}{5}(2x-1)$ are consecutive integers. What integers are they? Is there more than one answer?

Y. 8

1. Simplify (i) $\dfrac{x-4}{(x-2)(x+3)}+\dfrac{14}{(x-2)(x+3)(x+5)}$;

   (ii) $\dfrac{a^2-ab}{a^2-2a-3}\times\dfrac{a^2+2ab-3a-6b}{a^2+ab-2b^2}$.

2. The equation $s=a+bt+t^2$ is satisfied by the two pairs of values $t=3$, $s=5$ and $t=5$, $s=29$; a and b are constants, find their values.

3. It is known that $\dfrac{6a-7}{2}$ is greater than $\dfrac{5a+4}{3}$. If a is an integer, find its least value.

4. Solve (i) $\dfrac{1}{2x-1}=\dfrac{3x-4}{3x}$;

   (ii) $x=2y+3$, $x^2-3y^2=22$.

5. A man buys a number of motor-cars at £400 each. He keeps one for his own use and sells the remainder at 10 per cent. profit. As the result, he finds that he has got his own car for nothing. How many cars did he buy?

Y. 9

1. Prove that $x^2 + x - 1$ is a factor of $x^5 - 5x + 3$. Find a factor of $y^5 - 5y - 3$.

2. (i) Make μ the subject of the formula, $\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{R}\right)$.

   (ii) If $3x + y - 2z = 7$ and $x - y - 2z = 5$, find the *numerical* value of $2x + y - z$.

3. Solve (i) $W(x - 3) = 12$, $W(x - 2) = 14$.

   (ii) $3x^2 - 5x = 9$ (to 2 places of decimals).

4. (i) Form an equation having as roots, $-2\frac{1}{2}$, $\frac{3}{8}$.

   (ii) What can you say about c if $x^2 - 12x + c = 0$ has no roots?

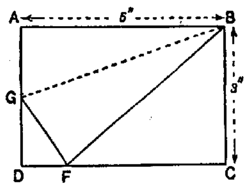


FIG. 22.

5. A rectangular sheet of paper ABCD (see Fig. 22) is folded so that A falls on F, with BG as crease. Show that DF = 1 inch. Also, if AG = x in., express GF, GD in terms of x; then obtain an equation in x and solve it.

Y. 10

1. (i) Factorise $ab(1 + c^2) - c(a^2 + b^2)$.

   (ii) If $x - y = 4$ and $x^2 - y^2 = 64$, prove that $x^3 - y^3 = 28^2$.

2. If $x = \frac{a-1}{a+1}$ and $y = \frac{2a-1}{2a+1}$, find a in terms of x. Find also an equation connecting x and y not involving a.

3. Two numbers R, v are connected by the equation,

$$R = a + b\sqrt{v},$$

where a, b are constants. If $R = 10$ when $v = 64$ and $R = 11{\cdot}2$ when $v = 81$, find the values of a, b.

4. Solve (i) $\frac{1}{2(x-1)} + \frac{3}{x^2 - 1} = \frac{1}{4}$;

   (ii) $xy = 40$, $yz = 60$, $x + z = 20$.

5. A boy is told to think of a number, subtract q from it and divide the result by r. By mistake he subtracts r and divides by q, but gets the same answer as he otherwise would have done. Find the number he obtains.

SUPPLEMENTARY TEST PAPERS Z. 1-10. (Part III, Ch. I-III)

Z. 1

1. Simplify :

(i) $4a^6 \div a^2$; (ii) $3b^{\frac{1}{3}} \div b^{\frac{1}{6}}$; (iii) $4c^{-3} \times 2c^{-2}$;
(iv) $6d^{\frac{2}{3}} \times d^{\frac{4}{3}}$; (v) $(2n^{\frac{1}{3}})^3$; (vi) $y^{-3} \times y^3$.

2. Find the value of

(i) $\sqrt{\left(\frac{\mathsf{HD}^5}{\mathsf{L}}\right)}$ when $\mathsf{H} = 18{\cdot}4$, $\mathsf{D} = 3{\cdot}06$, $\mathsf{L} = 880$;

(ii) $\mathsf{P}^{\frac{1}{2}}\mathsf{H}^{-\frac{3}{4}}$ when $\mathsf{P} = 144$, $\mathsf{H} = 16$.

3. Solve (i) $\frac{2}{x} + \frac{y}{2} = 1 = \frac{3}{x} + \frac{y}{3}$.

(ii) $(z-3)^4 = 16$.

4. Write as simply as possible :

(i) $\frac{3}{\sqrt{12}}$; (ii) $\sqrt{45} - \sqrt{20}$; (iii) $\frac{12}{\sqrt{5}-1}$.

5. The displacements of similar ships vary as the cubes of their lengths. Find the displacement of a ship which is half as long as a similar ship of displacement 2000 tons.

Z. 2

1. If $x = \sqrt{y}$, $y^3 = z^2$, $z = t^5$, express as powers of t, (i) y, (ii) x. Also express z as a power of x.

2. Find x, y, z if

(i) $x = 10^{1{\cdot}7796}$; (ii) $10^y = 0{\cdot}52$; (iii) $z^{-3} = 0{\cdot}078$.

Evaluate $\sqrt[3]{(0{\cdot}623)} \div (0{\cdot}472)^2$.

3. (i) Factorise $(x^2 - 5x)^2 + 10(x^2 - 5x) + 24$.

(ii) Solve $\frac{1}{x} + \frac{1}{y} = 5$, $6xy = 1$.

4. A certain sheet of transparent material absorbs $\frac{1}{10}$th of the red light falling on it. How many sheets must be placed one over another so that the red light is reduced by 60 per cent. ?

5. The weight of the shell of a d-inch gun (*i.e.* internal diameter, d in.) varies as d^3. Find the internal diameter of a gun whose shell is twice as heavy as the shell of a $4{\cdot}7$ in. gun.

Z. 3

1. (i) What are the values of $(10^{-3})^{\frac{1}{3}}$, $(10^{-4})^{\frac{1}{2}}$, $(0{\cdot}1)^{-3}$?
   (ii) If $V = \frac{4}{3}\pi r^3$, find r to 3 figures when $V = 1$.

2. Rewrite the following without using negative or fractional indices : (i) $(1+x)^{-3}$; (ii) ut^{-1} ; (iii) $l^{\frac{1}{2}}g^{-\frac{1}{2}}$.

3. (i) Solve $\dfrac{5x+2}{2x-1} - \dfrac{x-13}{x+3} = 4$.
   (ii) If $y = 4x+5$ and $6xz = 3y - 2x$, find z in terms of y.

4. Given $\sqrt{2} \simeq 1{\cdot}414$ and $\sqrt{3} \simeq 1{\cdot}732$, evaluate

$$\text{(i) } \frac{4}{\sqrt{2}}; \quad \text{(ii) } \frac{6}{\sqrt{3}}; \quad \text{(iii) } \frac{\sqrt{3}}{\sqrt{6}}; \quad \text{(iv) } \frac{1}{\sqrt{3}-\sqrt{2}}.$$

5. The electrical resistance of a wire varies as its length and inversely as the square of its diameter. Compare the resistances of two copper wires, one 100 ft. long, diameter $\frac{1}{16}$ inch, the other 50 ft. long, diameter $\frac{1}{10}$ inch.
State in words how the diameter varies with the length and the resistance.

Z. 4

1. Write as simply as possible :

$$\text{(i) } \frac{\sqrt{45}}{\sqrt{3}}; \quad \text{(ii) } \frac{\sqrt{5}}{\sqrt{20}}; \quad \text{(iii) } \frac{\sqrt{6}}{\sqrt{3}-\sqrt{2}}; \quad \text{(iv) } 4^{\frac{3}{4}} \times 2^{-\frac{1}{2}}.$$

2. The volume of a sphere, radius r in., is $\frac{4}{3}\pi r^3$ cu. in. ; find the volume of metal in a hollow spherical shell, if the external diameter is $11{\cdot}5$ in., and the metal is $0{\cdot}7$ in. thick.

3. (i) If $y = ax + \dfrac{b}{x^2}$ where a, b are constants, and if $y = -3$ when $x = 2$, and if $y = 6$ when $x = -1$, find y if $x = \frac{1}{2}$.
   (ii) Express $\dfrac{2x^2+9x-21}{x+5}$ in the form $px + q + \dfrac{r}{x+5}$.

4. If $x = cy^{1\cdot25}z^{-0\cdot5}$, express z in terms of c, x, y. Evaluate c to 2 figures if $z = 6{\cdot}4$ when $x = 11$ and $y = 18$.

5. An aeroplane travelling against a wind of 20 ft. per sec. takes half as long again to travel a certain distance as another aeroplane travelling with the wind. If there is no wind, the first takes 46 minutes and the second 44 minutes. What is the distance ?

Z. 5

1. Find the values of (i) $10^{-\frac{4}{5}}$; (ii) $(\frac{4}{5})^{-10}$. What is the least integral value of n for which $(0{\cdot}95)^n$ is less than $0{\cdot}01$?

2. Simplify $(2\sqrt{3}+3\sqrt{2})^2 - (\sqrt{6}-\sqrt{2})^2$.

3. Prove that $\frac{6}{2x^2-x}$ and $7-x(2x-1)$ are equal, both when $x=2$ and when $x=-\frac{1}{2}$. For what other values of x are they equal ?

4. What are the values of a, b if the following equations have the same roots for x :

$$x^2+1=ax+b\ ;\quad 3x^2-2=bx+8a\ ?$$

5. The square of the time taken by a planet to revolve round the Sun varies as the cube of its mean distance from the Sun. If the mean distances of Jupiter and the Earth from the Sun are in the ratio 16 : 3, find the number of our days in Jupiter's year.

Z. 6

1. Express 4^n and 8^{n-1} as powers of 2, and simplify

$$2^n\times 4^n\div 8^{n-1}.$$

2. Evaluate $e^{\mu\theta}$ when $e=2{\cdot}718$, $\mu=0{\cdot}35$, $\theta=1{\cdot}8$.

3. If $\sqrt{a}+\sqrt{b}=1$, prove that $(a-b+1)(a-b-1)=2(a+b-1)$.

4. (i) What is the ratio of the roots of $3x^2+7x=26$?

(ii) Solve $y^2-1=x(2x-1)$, $y-1=x$.

5. The force P lb. acting on a body is inversely proportional to the square of the distance, r ft., of the body from a fixed point. If to the square of the velocity v (in ft. per sec.) a certain fixed number is added, the result is inversely proportional to r. If $\mathsf{P}=2$ and $r=3$, then $v^2=7{\cdot}5$, and if $\mathsf{P}=\frac{8}{9}$, then $v^2=4{\cdot}5$. What is v^2 if $\mathsf{P}=4\frac{1}{2}$? Also express P in terms of v.

Z. 7

1. Evaluate $\frac{0{\cdot}045}{d^{1{\cdot}16}}\times\frac{v^2l}{2g}$ when $l=75$, $v=7{\cdot}4$, $d=2{\cdot}3$, $g=32{\cdot}2$.

2. (i) Simplify $\frac{1}{8^{-\frac{2}{3}}}-\frac{1}{16^{-\frac{3}{4}}}$.

(ii) Express in prime factors $18^n\times 12^{2n}\times 6^{3n}$.

3. (i) Make d the subject of the formula, $w=\mathsf{W}\sqrt{\left(\frac{\mathsf{D}^2+d^2}{2d^2}\right)}$.

(ii) Find x, y if $x=\frac{1}{y}=\frac{20}{x-1}$.

4. Find an integral value of x such that the expressions,

$$\frac{x}{2}+\frac{5}{6},\quad \frac{x+2}{4}+\frac{x}{3},\quad \frac{x+2}{2}$$

are in ascending order of magnitude.

5. A motor boat can travel at 12 m.p.h. in still water. It goes 15 miles down stream and returns against the current. If the whole journey takes 10 minutes more than it would do in still water, find the speed of the current.

Z. 8

1. Find to 3 figures the value of

(i) $4 \times \sqrt{1000} \div \sqrt[3]{100}$; (ii) $(0{\cdot}08)^{-0{\cdot}4}$.

2. Find the value of x if

(i) $27^x = 3$; (ii) $(\frac{1}{9})^x = 3$; (iii) $9^x = \frac{1}{3}$;
(iv) $3^x = (\frac{1}{3})^x$; (v) $\frac{1}{3}$ of $27^x = 9^{2x}$.

3. Simplify (i) $\sqrt{(42 + 12\sqrt{6})} + \sqrt{(42 - 12\sqrt{6})}$;
(ii) $\{a + \sqrt{(a^2 - b^2)}\}\{a - \sqrt{(a^2 - b^2)}\}$.

4. Solve the equations:

(i) $2x - 3y = 1$, $x^2 - 2x = y^2 - 1$.
(ii) $\sqrt{(2x + 3)} + \sqrt{(x - 2)} = 2\sqrt{(x + 1)}$.

5. The weight that can be supported by a cylindrical strut varies as the fourth power of the diameter of the strut and inversely as the square of its length. If one strut is twice as long as another, and if they carry equal weights, what should be the ratio of their diameters?

Find also how the length of a strut varies with its volume and the weight it can support.

Z. 9

1. The bore d in. of a pipe through which a pump of horse-power H can deliver G gallons per sec. is given by $d = 1{\cdot}25\mathrm{G}^{\frac{1}{2}}\mathrm{H}^{-\frac{1}{4}}$. (i) Find the bore necessary for an engine of 15 H.P. required to deliver 300 gallons per minute. (ii) Make G the subject of the formula, and express the numerical coefficient to 2 figures.

2. (i) Subtract $\dfrac{x^n}{(x+y)^m}$ from $\dfrac{x^{n-1}}{(x+y)^{m-1}}$;

(ii) Simplify $(a^{\frac{1}{3}} - b^{\frac{1}{3}})(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})$.

3. (i) Prove that $2x - y$ is a factor of $4x^3 - 3xy^2 + y^3$ and find the other factors.

(ii) Solve $x^2 + xy = 2$, $xy + 3y^2 = 1$.

4. If $(h - a)(1 - a) = (h - b)(1 - b) = k$, find k in terms of a, b.

5. The annual cost of running a car is the sum of two quantities, one of which is fixed and the other varies directly as the mileage. If the cost is £40 in a year when the mileage is 3000, and £47. 10s. when the mileage is 4500, find the cost when the mileage is 7000.

Z. 10

1. Prove that $2^{\sqrt{3}} - 2^{-\sqrt{3}}$ is nearly equal to 3.

2. What are the logarithms (to base 10) of 1000, $\sqrt{1000}$, $\sqrt[3]{100}$, 0·01, $\dfrac{1}{\sqrt{0 \cdot 1}}$?

Express 2 as a power of 5.

3. (i) Find x to 3 figures if $(x+1)^3 = 2x^3$.

(ii) The quadratics

$$x^2 + 81x - 1666 = 0 \text{ and } 2x^2 - 185x + 2567 = 0$$

have a common root. Find it.

4. The cube of the horse-power required for a ship of given type varies as the ninth power of the speed and the square of the displacement. A ship of 1000 tons displacement is found to require a certain H.P. at 10 knots; by what factor must this H.P. be multiplied for a similar ship of 8000 tons to give 12 knots?

5. I think of an odd number. I then multiply it by 3 and divide by 2 and find that the quotient is even. I then multiply the quotient by 3 and divide by 2, and obtain a number between 170 and 180. Find the original number.

Z. 11-20. (Part III, Ch. I-V)

Z. 11

1. Simplify (i) $20^{\frac{1}{2}} \times 5^{\frac{3}{2}}$; (ii) $(2 \cdot 25)^{-1 \cdot 5}$; (iii) $6^{4x} \div 18^{2x}$.

2. Prove that

$$x = 1 + \sqrt{2} \text{ is a root of the equation, } x^4 - 6x^2 + 1 = 0.$$

What are the other roots?

3. Sum to n terms,

$$\text{(i) } \tfrac{5}{6} - \tfrac{5}{9} + \tfrac{10}{27} - \dots; \quad \text{(ii) } \tfrac{5}{6} + \tfrac{7}{12} + \tfrac{1}{3} + \dots .$$

Which of these series has a "sum to infinity" and what is it?

4. If $pv^n = c$ where n, c are constants, and if $p = 90$ when $v = 4$, and if $p = 40$ when $v = 6 \cdot 2$, find the values of n and c. Find also the value of v when $p = 60$.

5. A cylindrical boiler is made with hemispherical ends. If its greatest length is a feet and if its breadth is b feet, show that its volume is $\dfrac{\pi}{12} b^2(3a - b)$ cu. feet.

Z. 12

1. Simplify (i) $(27^{\frac{2}{3}} + 9^{\frac{3}{2}}) \times 81^{-\frac{3}{4}}$;

(ii) $\dfrac{\sqrt{18}}{(1+\sqrt{3})(\sqrt{6}-\sqrt{2})}$.

2. If $a = \log \frac{6}{5}$, $b = \log \frac{10}{9}$, $c = \log \frac{25}{24}$, express as simply as possible $3a + b + c$.

3. (i) Sum to n terms the series

$$\log a + \log \frac{a}{b} + \log \frac{a}{b^2} + \dots .$$

(ii) Find to 2 significant figures the value of the sum of 20 terms of the series, $\frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \dots$.

4. Solve (i) $u + v = 6$; $u^2v^2 + 2uv = 35$;

(ii) $(3x - 1)^4 = (x + 3)^4$.

5. A series of similar articles is made in different sizes. Their weights form a series in A.P., and their values a series in G.P. Size 1 weighs 3 lb. and is worth 2s. 8d.; size 4 weighs $7\frac{1}{2}$ lb. and is worth 9s. Find the weight and value of size 7.

Z. 13

1. (i) Simplify $2^{n+3} - 6 \times 2^n$.

(ii) Find x if $4^x = 32^3$.

2. The population of a country increases by 3 per cent. every 10 years. What is the percentage increase in 70 years?
What is the value of n if $(1{\cdot}03)^n = 2$?

3. Factorise (i) $6x^2 - 11x - 10$; (ii) $a^2 + ab + b - 1$.
Divide $\{\log(a^2) - \log(b^2)\}$ by $\{\log a - \log b\}$.

4. (i) Sum the series

$(n - n^2), (2n - n^2), (3n - n^2), \dots$ to $2n$ terms.

(ii) Two consecutive terms of a H.P. are $\frac{2}{5}$ and $\frac{12}{23}$; what is the largest positive term in the series?

5. A non-stop train leaves A for B at 8 a.m., and another non-stop train leaves B for A at 8.15 a.m. The trains arrive at their destinations 20 minutes and 50 minutes respectively after passing one another. If the speed of each train is uniform, find the time at which they meet.

Z. 14

1. (i) Evaluate $\dfrac{3 \cdot 28 \times 10^{-9} \times \frac{5}{3}}{\frac{1}{4}\pi \times (7 \cdot 26 \times 10^{-3})^2}$.

   (ii) Express $18^{2x} \div 12^{3x}$ in the form $2^p \,.\, 3^q$.

2. If $a = 2\sqrt{2} - \sqrt{7}$ and $b = 2\sqrt{2} + \sqrt{7}$, simplify $ab^{-2} + ba^{-2}$.

3. Solve (i) $x + y = z = 2xy$; $\dfrac{1}{x} - \dfrac{1}{y} = 1$.

   (ii) $\sqrt{(x+4)} + \sqrt{(x-3)} = \sqrt{7}$.

4. (i) The 3rd term of an A.P. is 8, and the 10th term is 36. Find the sum of n terms.

   (ii) What is the least number of terms of the G.P., 4, 6, 9, ... whose sum is greater than 10,000 ?

5. Two men, A, B, receive the same initial salary, £200 a year. A receives an increase of £20 a year at the end of every 2 years; B receives an increase of £10 a year at the end of every year. Find the total sum each has received at the end of $2n$ years, n being an integer.

Z. 15

1. (i) Simplify $5 \times 4^{3n+1} - 20 \times 8^{2n}$.

   (ii) Find x if $\log_{10}(x+1) - \log_{10} x = 1$.

2. Find rational numbers a, b such that

$$(a + b\sqrt{2})(6 - \sqrt{2})^2 = 3 + \sqrt{2}.$$

3. Solve (i) $\frac{1}{2}x^2 + x = 3$ (to 2 places of decimals).

   (ii) $3x + 4y = 18$, $x^2 + 3xy + 2y^2 = 40$.

4. (i) Find the sum of all positive integers less than 1000 which are not divisible either by 2 or 5.

   (ii) Sum to n terms the series

$$1 + \frac{x}{2},\ 2 + \frac{x^2}{4},\ 3 + \frac{x^3}{8},\ 4 + \frac{x^4}{16},\ \dots .$$

5. Show that a triangle whose sides are of lengths $(\sqrt{5} - 1)$ in., $(\sqrt{5} + 1)$ in., $2\sqrt{3}$ in. is right-angled, and find its area.

Z. 16

1. (i) Simplify $(a^3b^{-1})^{-2} \times (a^{-3}b^3)^{-1}$;

   (ii) Evaluate $\dfrac{a}{b} + \dfrac{b}{a}$, if $a = 3\sqrt{2} - 2\sqrt{3}$ and $b = 3\sqrt{2} + 2\sqrt{3}$.

2. (i) Factorise $2x^2 - \dfrac{3}{x^2} + 1$.

   (ii) Simplify $(9^{2n} - 4^{2n}) \div (3^n - 2^n)$.

3. (i) Express $\log_2 6$ in the form $\log_8 n$.

(ii) Express, without using logarithmic notation, the relation, $\log_9 x = 3 \log_3 y$.

4. (i) Sum to $(2n+1)$ terms, n being integral, the series

$$1 - 4 + 7 - 10 + 13 - \ldots.$$

(ii) The 3rd term of an H.P. is 5, and the 5th term is 3. Find (i) the 15th term; (ii) the rth term.

5. The time for a railway journey varies directly as the distance and inversely as the velocity. The velocity varies directly as the square root of the amount of coal used per mile and inversely as the number of coaches in the train. For a journey of 25 miles in half an hour with 18 coaches, 10 cwt. of coal is consumed. How much coal is used in a journey of 15 miles in 20 minutes with 24 coaches?

Z. 17

1. Simplify (i) $4^{n+3} \times 8^{2-n} \div 16^{3-n}$; (ii) $(\frac{9}{16})^{-\frac{1}{2}} \times (3\frac{3}{8})^{\frac{1}{3}}$.

2. (i) Simplify $\{\sqrt{(x+1)} - \sqrt{(x-1)}\} \div \{\sqrt{(x+1)} + \sqrt{(x-1)}\}$.

(ii) If $\log \dfrac{b+c}{2} = \frac{1}{2}(\log b + \log c)$, prove that $b = c$.

3. Solve (i) $x^2yz = 12,\ xy^2z = 6,\ xyz^2 = 18$;

(ii) $\sqrt{(x-3)} + \sqrt{(x+2)} = \sqrt{(3x+4)}$.

4. (i) The sum of the first 5 terms of an A.P. is -2, and the sum of its first 9 terms is 0. Find the sum of the first n terms.

(ii) Use logarithms to evaluate the sum to 10 terms of the G.P., $3 + 2^{\frac{1}{4}} . 3^{\frac{3}{4}} + 2^{\frac{1}{2}} . 3^{\frac{1}{2}} + \ldots$.

5. If $r^x = 10$ and $r^{x+4} = 20$, what is the value of r^{x+8}? Find also the value of r.

Z. 18

1. (i) Multiply $(\sqrt{2} + \sqrt{3} + \sqrt{5})(\sqrt{2} - \sqrt{3} + \sqrt{5})$ by $(\sqrt{2} + \sqrt{3} - \sqrt{5})(-\sqrt{2} + \sqrt{3} + \sqrt{5})$.

(ii) Find x if $\dfrac{\log x}{\log a^2} = \dfrac{\log b^2}{\log b}$.

2. If $2^x = 5^y = 10^z$, prove that $z = \dfrac{xy}{x+y}$.

3. (i) Solve $3x + 5y = 4$; $\dfrac{3}{x} + \dfrac{5}{y} = 17$.

(ii) If $xy - ab = a(x+y)$, express $x+y$ in terms of x, a, b.

4. (i) The sum of n terms of a series is $3n^2 - 5n$ for all values of n. Find the 1st term, the 2nd term, the rth term.

(ii) What is the sum of the series,

$$1 + (a + a) + (a^2 + 2a) + (a^3 + 3a) + \dots + (a^n + na)\ ?$$

5. A man borrows £1400 to be repaid by 5 equal annual instalments, the first being made 12 months after the loan is contracted. Reckoning compound interest at $4\frac{1}{2}$ per cent. per annum, find the amount of each annual payment as accurately as your tables permit.

Z. 19

1. (i) Simplify $9^{2n+1} \times 6^{2n-3} \div (3^{5n-2} \times 18 \times 4^{n-2})$.

(ii) Find the square root of $57 - 12\sqrt{15}$.

2. Prove that $x = \sqrt{2} + \sqrt{3}$ is a root of $x^2 + \dfrac{1}{x^2} = 10$. What are the other roots ?

3. (i) Simplify $\log(\log x^p) - \log(\log x^q)$.

(ii) Find n if $4^{5n+2} = 8^{3n}$.

4. (i) Find the value of n if the sum of n terms of the series 2, 5, 8, 11, ... equals the sum of n terms of the series 47, 45, 43, 41, ...

(ii) The first term of a G.P. is x, its last term is x^2, and its sum is x^3. Find its common ratio.

5. The cost of material in a spoon of given design is the sum of two parts which vary as the square and the cube respectively of the length. If p shillings, q shillings are the costs for two spoons, the latter being twice as long as the former, prove that the cost for a spoon $1\frac{1}{2}$ times as long as the former is $\frac{9}{32}(4p + q)$ shillings.

Z. 20

1. (i) If $a + \dfrac{1}{a} = \sqrt{3}$, find the values of $a^2 + \dfrac{1}{a^2}$ and $a^3 + \dfrac{1}{a^3}$.

(ii) Simplify $\{b(b + c)^{-1} + c(b - c)^{-1}\}\{bc^{-1} + b^{-1}c\}^{-1}$.

2. (i) If $a^2 + b^2 = 14ab$, prove that $\log \dfrac{a + b}{4} = \frac{1}{2}(\log a + \log b)$.

(ii) Make n the subject of the formula, $T_1 = T_2\left(\dfrac{p_1}{p_2}\right)^{\frac{n-1}{n}}$, using logarithmic notation.

3. The equation $x^3 + ax^2 = bx - 1$, where a and b are rational, is satisfied by $x = 1 + \sqrt{2}$. Find a and b. Find also the other roots of the equation.

4. A man, whose income has increased every year by 5 per cent. of what it was the year before, made £1000 in the year 1901. What did he make in 1924, and what was his total income in the 24 years ?

5. If $s=1+3r+5r^2+7r^3+\ldots+(2n-1)r^{n-1}$, simplify the expression $s-rs$ as much as possible. Hence find s.

If $r^2<1$, find the sum to infinity of the given series.

Z. 21-30. (Part III, Ch. I-IX)

Z. 21

1. (i) Simplify $\dfrac{3^{n+3}-3\times 3^n}{4\times 3^{n+2}}$; (ii) Evaluate $(1\cdot 21)^{-\frac{1}{2}}$.

2. If $a=\frac{1}{2}(\sqrt{13}-1)$, prove that $\dfrac{5-a}{2-a}=4+a$ and evaluate $(2-a)(3+a)$.

3. (i) If $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}$, prove that $\dfrac{a}{d}=\dfrac{b(b^2-c^2)}{c(c^2-d^2)}$.

(ii) For what value of a is $x-1$ a factor of $3x^3+ax^2+4$, and what are the other factors in this case ?

4. The sum of the roots of $x^2-(c+3)x+2(c+1)=0$ is two-thirds of their product. Find the roots.

5. What digit must the symbol x represent if the product of the two numbers " $x5$ " and " $3x$ " is the number " $2xx5$ " ?

Z. 22

1. Solve $6^{2x-1}=8^x$ (2 places of decimals).

2. (i) If $\dfrac{a+x}{a-x}=\dfrac{y-b}{y+b}$, express y in terms of a, b, x.

(ii) If $\dfrac{a}{b}=\dfrac{c}{d}$, prove that $\dfrac{ac-c^2}{b-d}=\dfrac{d(a^2+c^2)}{b^2+d^2}$.

3. Show that $x^4+2x^3-3x^2-8x-4$ can be expressed in the form $(x^2+ax+2)(x^2+bx-2)$ and find the values of a, b.

Hence solve the equation, $x^4+2x^3-3x^2-8x=4$.

4. If s_r is the sum of r terms of the G.P., $1+3+9+27+\ldots$, find the sum of n terms of the series,

$$s_1+s_2+s_3+s_4+\ldots.$$

5. Prove that the sum of the cubes of three consecutive integers exceeds 3 times their product by 3 times their sum.

State and prove a similar result for three consecutive odd integers.

Z. 23

1. Simplify (i) $\log 12 + 2 \log 0{\cdot}75 - \log 6{\cdot}75$;

$$\text{(ii)}\ \frac{\sqrt{xy}}{\sqrt{x}+\sqrt{y}+\sqrt{(x+y)}}.$$

2. Find the square root of

$$x^6 - 4x^5 + 10x^4 - 10x^3 + 5x^2 + 6x + 1.$$

3. (i) If α, β are the roots of $x^2 - 4x + 1 = 0$, prove that $\alpha^3 + \beta^3 = 52$, and find the value of $\alpha^6 + \beta^6$.

(ii) If a is a root of $x^2 + px + q = 0$, prove that it is also a root of $x^3 - px^2 - (2p^2 - q)x = 2pq$.

4. Solve (i) $\dfrac{x}{a^2} = \dfrac{y}{b^2} = \dfrac{z}{c^2} = xyz$; (ii) $x - y = 3$, $x^3 - y^3 = 819$.

5. What is the coefficient of x^{10} in the expansion of

$$(x^2 - 1)(x+2)^{12}\ ?$$

Z. 24

1. Factorise (i) $2x^3 - 3x^2 - 3x + 2$; (ii) $a^2 - a(b+1) + 2(b-1)$.

2. If the graph of $y = ax^2 + bx + c$ passes through the points $(1, 7)$, $(2, 23)$, $(3, 17)$, find the values of a, b, c. Also find y when $x = 4$.

3. (i) If $ab + bc + ca = 0$, prove that $\dfrac{a}{b}$ is a square root of $\dfrac{a+c}{b+c}$.

(ii) What is the coefficient of x^4 in the expansion of

$$\left(3x^2 - \frac{1}{2x}\right)^8 ?$$

4. Find the sum of n brackets of the series,

$$(1+2),\ (1+2+4),\ (1+2+4+8),\ (1+2+4+8+16),\ \dots .$$

5. A train after running 1 hour is delayed for half an hour, after which its speed is reduced by one-quarter, and the train arrives at its destination $1\frac{1}{2}$ hours later than if it had travelled uniformly throughout. If the delay had occurred 45 miles

further on, the train would have been only 1 hour late. Find the distance travelled and the first speed of the train.

Z. 25

1. (i) If $8x^3 - 36x^2 + ax + b$ is a perfect cube, find the values of a, b.

(ii) If $a + \frac{1}{a} = 3$, evaluate $a^5 + \frac{1}{a^5}$.

2. (i) Find p, q if $x - 2$ and $x + 3$ are factors of $x^3 - px^2 - qx - 6$.

(ii) Factorise $4y^2 - (y^2 - z^2 + 1)^2$.

3. Find x if $(x+1)^2$ is greater than $5x - 1$ and less than $7x - 3$, and if x is an integer.

4. (i) Solve for x, $\frac{1}{x} - \frac{1}{x+a} = \frac{a}{b(a+b)}$.

(ii) Solve for x, y, z,

$$\frac{x+y}{2a} = \frac{y+z}{2b} = \frac{z+x}{2c} = yz.$$

5. Construct a quadratic in x, such that the arithmetic mean between its roots is A, and the geometric mean between them is G. Find also the harmonic mean between them.

Z. 26

1. Simplify $\sqrt{\{\frac{1}{2} - \frac{1}{3}\sqrt{2}\}}$.

2. Show how $4x^3 - 5x^2 + 6x - 7$ can be expressed in the form $a + b(x-1) + c(x-1)(x-2) + d(x-1)(x-2)(x-3)$, and find the numerical values of a, b, c, d.

3. (i) Factorise $x(y-z)^3 + y(z-x)^3 + z(x-y)^3$.

(ii) Simplify $\frac{(ax+by)^2 - (ay+bx)^2}{(cx+dy)^2 - (cy+dx)^2}$.

4. (i) Solve $\frac{x^2 - 2x + 5}{3x^2 + 4x - 1} = \frac{x^2 + 2x - 5}{3x^2 - 4x + 1}$.

(ii) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that $\frac{a^m}{b^{m-1}} + \frac{c^m}{d^{m-1}} + \frac{e^m}{f^{m-1}} = \frac{(a+c+e)^m}{(b+d+f)^{m-1}}$.

5. If both n and $\frac{5n+1}{7}$ are positive integers less than 10, prove that $\frac{11n+1}{9}$ is also a positive integer less than 10.

Z. 27

1. Simplify (i) $\dfrac{9^{n+2} - 12 \times 3^{2n+1}}{27^n}$; (ii) $\dfrac{\sqrt{(8 - 4\sqrt{3})}}{3 - \sqrt{3}}$.

2. (i) If $(a+b) : (a-b) = 7 : 4$, evaluate $(a^2+b^2) : (a^2-b^2)$.

   (ii) If y is a mean proportional between x and z, prove that
$$\frac{x^2+y^2+z^2}{x+y+z} = \frac{y^3+z^3}{z(y+z)}.$$

3. (i) Prove that the roots of $x^3 - 18x + 125 = 0$ are the cubes of the roots of $x^2 + 3x + 5 = 0$.

   (ii) For what values of x is $4x^2 + 5x$ greater than 6?

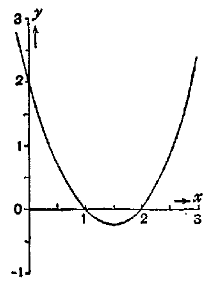


FIG. 23.

4. Fig. 23 represents the graph of $y = (x-1)(x-2)$. Sketch, with the same scale, the graphs of (i) $y = 1 + (x-1)(x-2)$; (ii) $y = (x+1)(x+2)$; (iii) $y = (x-2)(x-3)$; (iv) $y = \dfrac{1}{(x-1)(x-2)}$.

5. The price of gas is reduced by x per cent., and a household consequently consumes x per cent. more gas in a year. Prove that the yearly gas bill is certainly reduced. Find the value of x if the bill is reduced by 9 per cent.

Z. 28

1. (i) Factorise $x^4 + 2x^2 + 9$.

   (ii) Show that
$$(x+y+z)^4 - (y+z)^4 - (z+x)^4 - (x+y)^4 + x^4 + y^4 + z^4$$
is equal to $12xyz(x+y+z)$.

2. Express $0 \cdot 97\dot{2}$ as a series, and find the sum to infinity.

3. (i) Solve for x and y,

$$x(1+y)=b(a-1), \quad y(1+x)=a(b-1).$$

(ii) Find two pairs of solutions of the equations,

$$x^{x+y}=y^{24}, \quad y^{x+y}=x^{6}.$$

4. Prove that the relation $y=\frac{2(4x-5)}{x+1}$ can be expressed in the form $\frac{y-a}{y-b}=\frac{2(x-a)}{x-b}$, and find the numerical values of a, b.

5. What is the nth term of the series, $1, \frac{3}{2}, \frac{5}{4}, \frac{7}{8}, \ldots$?
If the sum of n terms is denoted by s, write down the series obtained by subtracting $\frac{1}{2}s$ from s, and then find its value in terms of n. Hence find s.
What is the sum to infinity of the given series ?

Z. 29

1. If $\frac{a+b}{b+c}=\frac{c+d}{d+a}$, prove that either $a=c$ or $a+b+c+d=0$.

2. Express $\mathsf{H}=\frac{\mathsf{G}^2\mathsf{L}}{80d^5}$ in the form $d=k \,.\, \mathsf{G}^p\mathsf{L}^q\mathsf{H}^r$, giving k correct to 2 significant figures.

3. (i) Factorise $x^2-xy-6y^2-3x+19y-10$.

(ii) Solve $x-6=y+4=z-3=\sqrt[3]{(xyz)}$.

4. For what value of k are the roots of the equation,

$$\frac{x^2+5x}{7x+9}=\frac{1-k}{1+k},$$

equal in magnitude and opposite in sign ?

5. Find the condition that the roots of

$$3x^2-6xy+10y-3=0$$

regarded as an equation in x, may be (i) real and different, (ii) coincident, (iii) imaginary.

Use these results to sketch the graph of $y=\frac{3(x^2-1)}{2(3x-5)}$.

Z. 30

1. a, b, c are in continued proportion, and are the mean proportionals to x and y, y and z, z and w respectively. Prove that $\frac{x}{y}=\frac{z}{w}$.

2. (i) What is the coefficient of x^3 in

(a) $(1-x^2)({}^{3}1+x)^8$; (b) $(1-x)^2(1+x)^8$?

(ii) Find r if the coefficients of x^r and x^{r+1} in the expansion of $(3x+2)^{19}$ are equal.

3. Solve $x^2=5x-y,\ y^2=5y-x$.

4. If s equals the sum of n terms of the series,

$$1+x(1+x)+x^2(1+x+x^2)+x^3(1+x+x^2+x^3)+\dots,$$

write as simply as possible the terms of the series $s(1-x)$.

Then prove that

$$s(1-x)(1-x^2)=(1-x^n)(1-x^{n+1}).$$

5. Find a quadratic equation such that the sum of the squares of the roots is greater by 40 than the sum of the roots, and the sum of the cubes of the roots is greater by 20 than the sum of the squares of the roots.

SUPPLEMENT

[*covering further requirements of certain examining bodies for "additional mathematics" in School Certificate*]

I. PERMUTATIONS AND COMBINATIONS

The r, s Principle

Example 1. There are 5 ways from A to B and 3 from B to C; how many are there from A to C via B?

Any one of the 5 ways from A to B can be combined with any one of the 3 ways from B to C.

$\therefore$ there are 5×3 different routes from A to C via B.

The principle used in this example may be stated as follows:

If one operation can be performed in r different ways, and if a second operation can then be performed in s different ways, the two operations can be performed in succession in $r \times s$ different ways.

Example 2. How many different arrangements can be made of the 4 letters, a, b, c, d?

Any one of the 4 letters can be put first. When the first place is filled, 3 letters remain, any one of which can be put second.

$\therefore$ the first two places can be filled in 4×3 ways.

2 letters remain, either of which can be put third, and 1 letter then remains for the fourth place.

$\therefore$ the number of different arrangements is $4 \times 3 \times 2 \times 1$.

This example illustrates the principle that if one operation can be performed in r different ways, and if a second operation can then be performed in s different ways, and if a third operation can then be performed in t different ways, the three operations can be performed in succession in $r \times s \times t$ different ways.

The principle can evidently be extended to any number of operations performed in succession.

Arrangements with Repetitions

Example 3. Residents at a boarding-house can choose *either* fish *or* eggs *or* bacon *or* sausages for breakfast. In how many ways can a man arrange his breakfasts for a week?

He has 4 choices on Sunday, 4 on Monday, 4 on Tuesday, and so on.

$\therefore$ the number of different arrangements for a week is

$$4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^7.$$

The number of arrangements of n unlike things, r at a time, when each may be repeated any number of times, is $\mathbf{n^r}$.

Imagine r compartments in a row, each of which can hold one of the unlike things, but no more.

The first compartment can be filled in n ways. And, as repetitions are allowed, the second can also be filled in n ways, and so on.

$\therefore$ the number of different arrangements is

$$n \times n \times n \times \dots r \text{ factors} = \mathbf{n^r}.$$

Factorials

The index notation makes it possible to write the result of Ex. 3 in a concise form. It is often desirable to have a short way of expressing the product of a number of consecutive integers. The product of the first n positive integers, $1 \times 2 \times 3 \times 4 \times \dots \times n$ is denoted by $n!$ or $\underline{|n}$ and is called "**factorial n.**"

Thus the answer to Example 2, p. 1A, could be given as $4!$. The product of any number of consecutive integers can be represented as the quotient of two factorials.

Thus $$9 \times 8 \times 7 \times 6 = \frac{9!}{5!}.$$

More generally, the product of r consecutive integers, of which n is the greatest, can be written

$$n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}.$$

Arrangements of Unlike Things in Line

The number of ways of arranging n unlike things all in a row is n!

Any one of the n things can be put first. When the first place has been filled, $(n-1)$ things remain, any one of which can be put second.

$\therefore$ the first two places can be filled in $n(n-1)$ ways.

When the first two places have been filled, $(n-2)$ things remain, any one of which can be put third.

$\therefore$ the first three places can be filled in $n(n-1)(n-2)$ ways; and so on.

$\therefore$ the total number of ways of arranging the n unlike things in a row is

$$n(n-1)(n-2)(n-3)\ldots 3.2.1=n!.$$

Circular Arrangements of Unlike Things

The number of ways of arranging n unlike things round a circle, regarding clockwise and counterclockwise arrangements as different, is (n − 1)!.

Since the order round the circle is all that matters, we can choose one special thing and keep it always in the same place. The remaining $(n-1)$ things can then be arranged in $(n-1)!$ ways.

If no distinction is made between clockwise and counterclockwise arrangements, the total number is half this amount, namely

$$\frac{(n-1)!}{2}.$$

Thus 6 men can be arranged at a round table in $5!$ ways; but if 6 beads of unlike colours are threaded on a ring, there are only $\frac{5!}{2}$ different designs.

Example 4. In how many ways can 10 different books be arranged on a shelf so that two particular books are next to one another?

Imagine the two particular books fastened together; this can be done in 2 ways, as either may come first.

There are now 9 unlike things to be arranged in a row, and this can be done in $9!$ ways.

$\therefore$ the number of arrangements is $9! \times 2$.

EXERCISE I. a

1. In how many ways can a boy and girl be chosen from 6 boys and 9 girls ?

2. In how many ways can 5 boys take their places on a bench ?

3. In how many ways can 3 different prizes be awarded to 10 boys, if any boy may win them all ?

4. In how many ways can a first and second prize be awarded in a class of 10 boys ?

5. In how many ways can a first, second and third prize be awarded in a class of 10 boys ?

6. In how many ways can I choose 1 novel, 1 magazine and 1 newspaper from 12 novels, 5 magazines and 8 newspapers ?

7. How many code words of 4 letters can be formed from the 26 letters of the alphabet ?

8. There are 5 gramophone records with a dance tune on each side of each record. In how many orders can the tunes be played, none being repeated ? How many arrangements are possible if there is only time to play 4 tunes ?

9. Find the values of

$$\text{(i) } 6!; \quad \text{(ii) } \frac{7!}{4!}; \quad \text{(iii) } \frac{8!}{5!\,3!}.$$

10. Express in factorials :

(i) $10 \times 9 \times 8$; (ii) $10 \times 11 \times 12 \times 13$;

(iii) $n(n-1)(n-2)$; (iv) $n(n+1)(n+2)(n+3)$;

(v) $n(n^2-1)(n^2-4)(n^2-9)$;

(vi) $n(n+1)(n+2)(n+3)\dots(n+r)$.

11. In how many orders can the letters A, B, C, D, E be marked round a circle drawn on the blackboard ?

12. Every day can be described as either fine or wet or indifferent. Within how many years must there be a repetition of the description of a week's weather ? (A week begins on Sunday.)

13. In how many orders can the letters of the word *treason* be arranged ? How many arrangements begin with *t* ? How many begin with *t* and end with *n* ?

14. In how many ways can 8 boys be arranged in a row ? In how many of these ways do 2 particular boys occupy end places ?

15. In how many ways can 6 people be arranged at a round table so that 2 particular people sit together ?

16. How many even numbers of 3 digits can be formed with the figures 4, 5, 6, (i) if no figure is repeated, (ii) if repetitions are allowed ?

17. Simplify (i) $n! \div (n-1)!$; (ii) $n! - (n-1)!$

18. In how many orders can the letters of the word *Sunday* be arranged? How many of these arrangements do not begin with *s*? How many begin with *s* and do not end with *y*?

19. There are n stations on a local railway line. How many different kinds of single third-class tickets must be printed if it is possible to book from any one station to any other?

20. Prove that $(2n)! \div n! = 2^n . \{1 . 3 . 5 . 7 \ldots (2n-1)\}$.

21. Prove that $2 . 6 . 10 . 14 \ldots (4n-6)(4n-2)$

equals $(n+1)(n+2)(n+3) \ldots (2n-1)2n$.

22. If $(mn+1)$ pairs of numbers are written down, each pair consisting of one chosen from the m letters $a_1, a_2, \ldots a_m$, and one from the n letters $b_1, b_2, \ldots, b_n$, prove that at least two of the pairs are identical.

Notation for Number of Permutations

Example 5. How many arrangements can be made with the 26 letters of the alphabet, if each contains 3 different letters?

Imagine three compartments in a row.

First letter	Second letter	Third letter

Any one of the 26 letters can be put in the first compartment. When this has been done, 25 letters remain, any one of which can be put in the second compartment.

$\therefore$ the first two compartments can be filled in 26×25 ways.

24 letters now remain, any one of which can be put in the third compartment.

$\therefore$ the total number of arrangements is $26 \times 25 \times 24$.

This is called "the number of **permutations** of 26 things, taken 3 at a time," and is written ${}_{26}P_3$.

More generally, the number of arrangements that can be made with n *unlike* things, if each arrangement contains r of them, is called **the number of permutations of n things, taken r at a time,** and is denoted by ${}_nP_r$.

In particular, ${}_nP_n$ denotes the number of ways of arranging n unlike things, all being taken; this is the number of ways of arranging n unlike things, all in a row, which was shown on p. 3A to be $n!$.

The number of arrangements, or permutations, of n unlike things, taken r at a time, is

$$_{n}P_{r}=n(n-1)(n-2)\dots(n-r+1).$$

The number of arrangements equals the number of ways of putting one thing into each one of a row of r compartments, when n unlike things are available.

The first compartment can be filled in n ways. When this has been done, the second can be filled in $(n-1)$ ways.

$\therefore$ the first two can be filled in $n(n-1)$ ways.

The third can then be filled in $(n-2)$ ways; therefore the first three can be filled in $n(n-1)(n-2)$ ways, and so on.

The rth or last compartment can finally be filled in $n-(r-1)$ ways, that is $(n-r+1)$ ways.

$$\therefore\ {}_{n}P_{r}=n(n-1)(n-2)\dots(n-r+1).$$

${}_{n}P_{r}$ has been defined only for $r=1, 2, 3, \dots, n$, and except for $r=n$, the expression just obtained for it can be written

$$_{n}P_{r}=\frac{n(n-1)(n-2)\dots(n-r+1)\times(n-r)!}{(n-r)!}=\frac{n!}{(n-r)!}.$$

Since ${}_{n}P_{n}=n!$, this result holds also for $r=n$ if $0!$ is given the value 1; the symbol $0!$ has not been previously defined in this book, and so we now define it to be 1.

Conditional Arrangements

In a few of the examples of Exercise I. a, the required arrangements were subject to very simple restrictions. The following example illustrates the procedure when the conditions are a little more elaborate.

Example 6. How many numbers greater than 7000 can be formed with the digits 3, 5, 7, 8, 9, no digit being repeated?

If the number contains 5 digits, it can be formed in ${}_{5}P_{5}=5!=120$ ways.

If the number contains 4 digits, the left-hand digit can be 7 or 8 or 9, but not 3 or 5; therefore the left-hand digit can be chosen in 3 ways. Whichever left-hand digit is chosen, the arrangement can be completed in ${}_{4}P_{3}$ ways.

$\therefore$ the number of arrangements with 4 digits is

$$_4P_3 \times 3 = 4 . 3 . 2 . 3 = 72.$$

$\therefore$ the total number of arrangements is $120 + 72 = 192$.

Like and Unlike Things

The number of ways of arranging n things, all in a row, when there are p alike of one kind, q alike of another kind, r alike of another kind, and so on, is

$$\frac{n!}{p!\, q!\, r! \ldots}.$$

Consider the n unlike letters,

$$a_1, a_2, a_3, \ldots, a_p, b_1, b_2, \ldots, b_q, c_1, c_2, \ldots, c_r, \ldots.$$

The number of ways of arranging n unlike things, all in a row, is $n!$.

Now if any one arrangement is written down, the letters, $a_1, a_2, \ldots, a_p$, can be arranged among themselves, *without altering the positions of any other letters*, in $p!$ ways. But if the suffixes of all the letters a are removed, these $p!$ different arrangements become identical.

$\therefore$ the number of arrangements with p like letters a and all other letters unlike, n letters in all, is $\frac{n!}{p!}$.

Similarly, in any one of these $\frac{n!}{p!}$ arrangements, the letters, $b_1, b_2, \ldots, b_q$, can be arranged among themselves, without altering the position of any other letters in $q!$ ways, and if the suffixes of all the letters b are removed, these $q!$ different arrangements become identical.

$\therefore$ the number of arrangements with p like letters a and q like letters b, and all other letters unlike, is $\frac{n!}{p!\, q!}$.

This argument can be repeated as often as necessary.

$\therefore$ the final number of arrangements is $\frac{n!}{p!\, q!\, r! \ldots}$.

Example 7. Find the number of ways of arranging the letters *aaaaabbbccdef* in a row, if the letters b are separated from one another.

The 10 letters *aaaaaccdef* can be arranged in $\frac{10!}{5!\, 2!}$ ways.

In any one of these arrangements, there are 11 places where the letter b_1 can be inserted. When this has been done, there are 10 places where b_2 can be inserted so as not to be next to b_1; and then 9 places where b_3 can be inserted so as not to be next to b_1 or b_2.

$\therefore$ the total number of arrangements of $aaaaaccdefb_1b_2b_3$ when b_1, b_2, b_3 are separated from one another is

$$\frac{10!}{5!\,2!} \times 11 \times 10 \times 9.$$

$\therefore$ when the suffixes are removed, the number is

$$\frac{10!}{5!\,2!} \cdot \frac{11 \times 10 \times 9}{3!} = \frac{10!\,11!}{2!\,3!\,5!\,8!}.$$

EXERCISE I. b

1. How many arrangements can be made with the letters, a, b, c, d, e, f, if each contains (i) 2 unlike letters, (ii) 5 unlike letters ?

2. A shelf holds 6 books. In how many ways can it be filled if 10 unlike books are available ?

3. How many numbers of 3 different digits can be formed with the figures 1, 2, 3, 4, 5, 6, 7 ?

4. How many numbers greater than 500 can be formed with the digits 4, 5, 6, 7, repetitions not being allowed.

5. In how many ways can 7 people be arranged at a round table so that the oldest and youngest sit together ?

6. In how many ways can 8 beads of different colours be arranged on a ring ? In how many of these arrangements are the red and yellow beads separated ?

7. Find the values of

$$\text{(i) } {}_5P_2\,; \quad \text{(ii) } {}_6P_3\,; \quad \text{(iii) } {}_5P_5.$$

8. Write down expressions for

$$\text{(i) } {}_nP_2\,; \quad \text{(ii) } {}_nP_5\,; \quad \text{(iii) } {}_{2n}P_n.$$

9. How many even numbers of 4 digits can be formed from the figures 2, 3, 4, 6, if repetitions are allowed ?

10. How many odd numbers above 4000 can be formed from the figures 1, 2, 3, 4, 6, if repetitions are not allowed ?

11. In how many ways can 5 dots and 3 dashes be arranged in a row ?

12. In how many ways can the letters in *rearrange* be arranged ? In how many of these do the letters *a* come together ?

13. In how many ways can 4 red counters, 4 white counters and 1 black counter be arranged in a row ?

14. In how many ways can 4 boys and 3 girls be arranged in a line so that boys and girls are placed alternately ?

15. In how many ways can the crew of an eight-oared boat be arranged (i) if 4 of the crew can row only on the stroke side, (ii) if 3 of the crew can row only on the stroke side and if 2 can row only on the bow side ?

16. In how many orders can 8 stories be arranged in a book if neither the longest nor the shortest comes first ? In how many of these ways does the longest come last ?

17. How many arrangements can be made of the letters in *photograph* ? In how many of these are there exactly 5 letters between the two letters *h* ?

18. Four travellers arrive at a town where there are 5 hotels. In how many ways can they be lodged (i) if no two go to the same hotel, (ii) if a given pair go to the same hotel, and the others to any of the other hotels, (iii) if there are no restrictions ?

19. In how many ways can 5 men and 2 ladies be arranged at a round table if the two ladies (i) sit together, (ii) are separated ?

20. In how many ways can 5 different Latin books, 4 different Greek books, 3 different French books be arranged on a shelf so that the books in each language come together ?

21. In how many ways can the letters in *deposit* be arranged if the vowels come in the even places ?

22. How many arrangements of the letters in *tomato* are such that the *t*'s are separated ?

23. Find the number of ways in which 10 candidates can be ranked in order of merit if (i) A is next to B, (ii) A is above B ? None are bracketed equal.

24. In how many ways can the 8 seats in a railway carriage be assigned to 8 men, 2 of whom must face the engine and 1 must have his back to the engine ?

25. In how many ways can 6 ladies and 6 gentlemen be arranged at a round table, if two particular ladies must not sit next to one particular man, all the men being separated ?

26. There are 5 *a*'s, 4 *b*'s, 3 *c*'s, 3 *d*'s, and 6 other different letters. In how many ways can they be arranged so that no two *a*'s come together ?

27. In how many different orders can 10 examination papers be set so that no two of the three mathematical papers are consecutive ?

Selections

In calculating the number of selections which can be made from a given set of unlike things, no regard is paid to the order in which the things occur in the chosen group. A change of order of the things in the group gives a new *arrangement*, but does not affect the *selection*.

There are two methods of tackling the general problem: the first treats it *ab initio*; the second is easier but assumes the formula obtained for ${}_nP_r$.

First Method

Example 8. How many selections of 2 letters can be made from the 6 letters, a, b, c, d, e, f?

If we write down all the different selections, each letter (such as a) comes 5 times in the list, namely once with each of the other letters (b, c, d, e, f); hence the total number of letters in the list is 6×5, and therefore the number of selections is $\frac{6 \times 5}{2}$.

This is called the number of **combinations** of 6 things taken 2 at a time and is denoted by ${}_6C_2$ or $\binom{6}{2}$.

Thus $${}_6C_2 = \frac{6 \times 5}{2}.$$

More generally, the number of ways of selecting r things from n unlike things is called the **number of combinations of n things, taken r at a time**, and is denoted by ${}_nC_r$ or $\binom{n}{r}$.

Example 9. How many selections of 3 letters can be made from the 7 letters, a, b, c, d, e, f, g?

If all the ${}_7C_3$ selections are written down, there are $3 \times {}_7C_3$ letters in the list, since each selection contains 3 letters. Each letter (such as a) comes ${}_6C_2$ times in the list since it occurs with each selection of 2 letters from the other 6 letters (b, c, d, e, f, g). Hence the total number of letters in the list is also $7 \times {}_6C_2$.

$$\therefore\ 3 \times {}_7C_3 = 7 \times {}_6C_2; \quad \therefore\ {}_7C_3 = \tfrac{7}{3} \times {}_6C_2.$$

But ${}_6C_2 = \frac{6 \times 5}{2}$, see Example 8,

$$\therefore\ {}_7C_3 = \frac{7}{3} \times \frac{6 \times 5}{2} = \frac{7.6.5}{1.2.3}.$$

The number of selections, or combinations, of n unlike things, taken r at a time, is

$$_n\mathrm{C}_r=\frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r}=\frac{n!}{r!(n-r)!}.$$

Denote the n unlike things by the letters, $a_1, a_2, a_3, \dots, a_n$.

If all the $_n\mathrm{C}_r$ selections are written down, there are $r\times{}_n\mathrm{C}_r$ letters in the list, since each selection contains r letters. Each letter (such as a_1) comes $_{n-1}\mathrm{C}_{r-1}$ times in the list since it occurs with each selection of $(r-1)$ letters from the other $(n-1)$ letters, $(a_2, a_3, \dots, a_n)$. Hence the total number of letters in the list is also $n\times{}_{n-1}\mathrm{C}_{r-1}$.

$$\therefore\ r\times{}_n\mathrm{C}_r=n\times{}_{n-1}\mathrm{C}_{r-1};\quad \therefore\ {}_n\mathrm{C}_r=\frac{n}{r}\times{}_{n-1}\mathrm{C}_{r-1}.$$

This relation holds for all values of n and r so long as $n\geqslant r>1$.

$$\therefore\ {}_{n-1}\mathrm{C}_{r-1}=\frac{n-1}{r-1}\times{}_{n-2}\mathrm{C}_{r-2};\quad {}_{n-2}\mathrm{C}_{r-2}=\frac{n-2}{r-2}\times{}_{n-3}\mathrm{C}_{r-3};$$

and so on, down to $_{n-r+2}\mathrm{C}_2=\dfrac{n-r+2}{2}\times{}_{n-r+1}\mathrm{C}_1$.

$$\therefore\ {}_n\mathrm{C}_r=\frac{n}{r}\cdot\frac{n-1}{r-1}\cdot\frac{n-2}{r-2}\dots\frac{n-r+2}{2}\times{}_{n-r+1}\mathrm{C}_1.$$

But $_{n-r+1}\mathrm{C}_1=n-r+1$, because 1 thing can be selected from $(n-r+1)$ things in $(n-r+1)$ ways.

$$\therefore\ {}_n\mathrm{C}_r=\frac{n(n-1)(n-2)\dots(n-r+2)(n-r+1)}{r(r-1)(r-2)\dots 2.1}.$$

Multiply numerator and denominator by $(n-r)!$,

$$\therefore\ {}_n\mathrm{C}_r=\frac{n!}{r!(n-r)!}.$$

Second Method

Example 10. How many selections of 3 letters can be made from the 7 letters, a, b, c, d, e, f, g?

Each selection of 3 letters can be arranged in 3! ways; for example, the selection *bcf* corresponds to the 6 different arrangements *bcf, bfc, cbf, cfb, fbc, fcb.*

But the number of arrangements of 7 things, taken 3 at a time, is $_7\mathrm{P}_3=7.6.5$. Therefore the number of selections is

$$_7\mathrm{P}_3\div 3!=\frac{7.6.5}{1.2.3},$$

and we write $\qquad _7\mathrm{C}_3={}_7\mathrm{P}_3\div 3!=\dfrac{7.6.5}{1.2.3}=\dfrac{7!}{3!\,4!}.$

The number of selections, or combinations, of n unlike things, taken r at a time, is

$$_n\mathrm{C}_r=\frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r}=\frac{n!}{r!(n-r)!}.$$

Each selection of r unlike things can be arranged in $r!$ ways; therefore each selection corresponds to $r!$ arrangements.

But the number of arrangements is

$$_n\mathrm{P}_r=n(n-1)(n-2)\dots(n-r+1)=\frac{n!}{(n-r)!}.$$

Therefore the number of selections is $_n\mathrm{P}_r\div r!$.

$$\therefore\ _n\mathrm{C}_r=\frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r}=\frac{n!}{r!(n-r)!}.$$

Since the number of selections must be an integer, the formula for $_n\mathrm{C}_r$ proves that $n(n-1)(n-2)\dots(n-r+1)$ is always a multiple of $r!$, that is to say, the product of any r consecutive integers is a multiple of $r!$.

The value of $_n\mathrm{C}_n$ is 1, since there is only 1 way of selecting n things from n things; thus, using the definition of $0!$ on p. 6, we see that the relation $_n\mathrm{C}_r=\dfrac{n!}{r!(n-r)!}$ holds even for $r=n$. It will also hold for $r=0$, if we define $_n\mathrm{C}_0$ to be 1, and this definition is also suggested by the important relation, $_n\mathrm{C}_r={}_n\mathrm{C}_{n-r}$, which we now proceed to prove.

There are two relations which deserve special attention:

$$\text{(i)}\quad {}_n\mathrm{C}_r={}_n\mathrm{C}_{n-r}.$$

If r things are selected from n things, $(n-r)$ things remain. Therefore the number of selections of r things from n, namely $_n\mathrm{C}_r$, is the same as the number of selections of $(n-r)$ things from n, namely $_n\mathrm{C}_{n-r}$.

Alternatively, we may say

$$_n\mathrm{C}_r=\frac{n!}{r!(n-r)!}$$

and

$$_n\mathrm{C}_{n-r}=\frac{n!}{(n-r)![n-(n-r)]!}=\frac{n!}{(n-r)!\,r!}.$$

(ii) ${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$.

${}_{n+1}C_r$ is the number of ways of selecting r letters from the $(n+1)$ unlike letters $a_1, a_2, a_3, \ldots, a_n, b$.

If each selection is made only from the letters a, the number of selections is ${}_nC_r$.

If each selection contains the letter b, it contains $(r-1)$ of the n letters a, and can therefore be made in ${}_nC_{r-1}$ ways.

$\therefore$ the total number of selections is ${}_nC_r + {}_nC_{r-1}$.

But the total number is ${}_{n+1}C_r$; $\therefore$ ${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$.

Alternatively, we may say

$$\begin{aligned} {}_nC_r + {}_nC_{r-1} &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} \\ &= \frac{n!\{(n-r+1)+r\}}{r!(n-r+1)!} = \frac{n!(n+1)}{r!(n-r+1)!} \\ &= \frac{(n+1)!}{r!(n-r+1)!} = {}_{n+1}C_r. \end{aligned}$$

Example 11. In how many ways can a committee of 3 women and 4 men be chosen from 8 women and 7 men? What is the number of ways if Miss X refuses to serve if Mr. Y is a member?

(i) There are ${}_8C_3$ ways of selecting 3 women from 8 women, and ${}_7C_4$ ways of selecting 4 men from 7 men.

$\therefore$ the number of ways of choosing the committee is

$${}_8C_3 \times {}_7C_4 = \frac{8.7.6}{1.2.3} \times \frac{7.6.5.4}{1.2.3.4} = 1960.$$

(ii) If both Miss X and Mr. Y are members, there remain to be selected 2 women from 7 women, and 3 men from 6 men. This can be done in ${}_7C_2 \times {}_6C_3 = \frac{7.6}{1.2} \times \frac{6.5.4}{1.2.3} = 420$ ways.

$\therefore$ the number of ways of making selections which do not include both Miss X and Mr. Y is $1960 - 420 = 1540$.

EXERCISE I. c

1. In how many ways can 3 books be selected from 7 different books?

2. In how many different ways can a football eleven be chosen from 14 boys?

3. In how many different ways can a hand of 3 cards be selected from a pack of 52 cards?

4. Find the number of ways in which a pair of triangles can be drawn with 6 given points as vertices, no three of the points being collinear.

5. What is the greatest number of points of intersection of (i) 12 straight lines, (ii) 9 circles, (iii) 6 straight lines and 5 circles ?

6. In how many ways can a committee of 4 men and 3 ladies be formed from 10 men and 8 ladies ?

7. There are 9 houses in a row. In how many ways can 5 of the front-doors be chosen for painting green ?

8. I have written 10 letters, but only have enough stamps for 4 of them. In how many ways can I choose the letters to be stamped ?

9. In how many ways can 10 men be divided into two groups of 3 and 7 respectively ?

10. In how many ways can 2 photographs and 3 drawings be selected from 5 photographs and 5 drawings ?

11. In how many ways can a committee of 5 be chosen from 10 candidates (i) so as to include both the youngest and oldest, (ii) so as to exclude the youngest if it includes the oldest ?

12. In how many ways can 3 books be chosen from a shelf holding n different books ? In how many of these ways is the longest book included ?

13. (i) n points are marked on a circle. How many chords can be obtained by joining them in pairs ?

(ii) How many diagonals does an n-sided polygon have, if a diagonal means any line joining two non-consecutive corners ?

14. In a group of 15 boys, there are 7 boy-scouts. In how many ways can 12 boys be selected so as to include (i) exactly 6 boy-scouts, (ii) at least 6 boy-scouts ?

15. In how many ways can a committee of 5 be chosen from 7 Conservatives and 4 Socialists so as to give a Conservative majority, if at least 1 Socialist is included ?

16. A committee of 6 is chosen from 10 men and 7 women so as to contain at least 3 men and 2 women. How many different ways can this be done if two particular women refuse to serve on the same committee ?

17. The results of 21 football matches (win, lose or draw) are to be predicted. How many different forecasts can contain exactly 18 correct results ?

18. Find n if (i) ${}_{n}C_{3} = {}_{n}C_{5}$; (ii) ${}_{n}C_{2} = 55$.

19. Simplify (i) ${}_{n+1}C_{3} \div {}_{n}C_{2}$; (ii) ${}_{n+1}C_{r+1} \div {}_{n}C_{r}$.

20. Simplify ${}_{4n}C_{3n} \times {}_{3n}C_{2n} \times {}_{2n}C_{n}$.

21. Prove that ${}_nC_r = \left(\frac{n+1}{r} - 1\right) \cdot {}_nC_{r-1}$.

22. Prove that ${}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}$.

23. Prove that the product of the first n odd numbers equals $(\frac{1}{2})^n \cdot {}_{2n}C_n \cdot {}_nP_n$.

Distribution in Groups

The number of ways of dividing (p + q + r) unlike things into 3 unequal groups, containing respectively p, q, r things, is

$$\frac{(p+q+r)!}{p!\,q!\,r!}.$$

A group of p things can be selected in ${}_{p+q+r}C_p$ ways. From the remaining $(q+r)$ things, a group of q things can be selected in ${}_{q+r}C_q$ ways. This leaves r things for the third group. Therefore the total number of ways is

$$\begin{aligned}&{}_{p+q+r}C_p \times {}_{q+r}C_q\\ &= \frac{(p+q+r)!}{p!\,(q+r)!} \times \frac{(q+r)!}{q!\,r!} = \frac{(p+q+r)!}{p!\,q!\,r!}.\end{aligned}$$

If two or more groups contain equal numbers of things, and if no regard is paid to the order of the groups, the number of different distributions is modified. Thus if $3p$ unlike things are divided into 3 groups each containing p things, the groups can be arranged in 3! orders. Therefore the number of possible distributions, if no regard is paid to the order of the groups, is

$$\frac{(3p)!}{p!\,p!\,p!} \div 3!.$$

Example 12. (i) How many different hands can be held by 4 men playing bridge (13 cards each)?

(ii) How many ways can a pack of 52 cards be arranged in 4 heaps of 13 cards each?

In (i), regard must be paid to the order in which the hands are placed on the table. The number of ways is

$${}_{52}C_{13} \times {}_{39}C_{13} \times {}_{26}C_{13} = \frac{52!}{(13!)^4}.$$

In (ii), no regard is paid to the order of the heaps. Therefore the number of ways is $\frac{52!}{(13!)^4} \div 4!$.

Selections taking any Number at a Time

The number of selections from n unlike things, taking any number at a time, is $2^n - 1$.

Each thing may be selected or rejected, that is, it may be disposed of in two ways.

$\therefore$ the number of ways of disposing of the n things is

$$2 \times 2 \times 2 \times \dots n \text{ factors} = 2^n.$$

But this includes the case where all are rejected.

$\therefore$ the number of selections, if at least one thing is chosen, is $2^n - 1$.

The total number may also be obtained by considering in succession groups containing 1, 2, 3, 4, ..., n things. The number of selections is therefore

$$_nC_1 + {_nC_2} + {_nC_3} + \dots + {_nC_n}.$$

It follows that

$$_nC_1 + {_nC_2} + {_nC_3} + \dots + {_nC_n} = 2^n - 1.$$

A different proof of this relation is given on p. 28A.

Given k unlike things and in addition p things alike of one kind, q alike of another kind, r alike of another kind, etc., the number of selections which can be made, taking any number at a time, is

$$2^k(p+1)(q+1)(r+1)\dots - 1.$$

From the p like things we can select either 0 or 1 or 2 or ... or p; we can therefore dispose of them in $(p+1)$ ways. Similarly the other groups of like things can be disposed of in $(q+1)$ ways, $(r+1)$ ways, etc.

Also each of the k unlike things can be disposed of in 2 ways (taken or left).

$\therefore$ the number of ways of disposing of them all is

$$(p+1)(q+1)(r+1)\dots(2 \times 2 \times 2 \times \dots k \text{ factors})$$
$$= 2^k(p+1)(q+1)(r+1)\dots.$$

But this includes the case where all are rejected.

$\therefore$ the number of selections, taking at least 1 thing, is

$$2^k(p+1)(q+1)(r+1)\dots - 1.$$

Example 13. How many different sums of money can be made up from five £1 notes, one 10s. note, four florins, three sixpences, one penny ?

The number is $(5+1)(1+1)(4+1)(3+1)(1+1)-1$

$$=6.2.5.4.2-1=480-1=479.$$

The value of r for which $_nC_r$ is greatest is $r=\frac{1}{2}n$ if n is even, and is $r=\frac{1}{2}(n-1)$ or $r=\frac{1}{2}(n+1)$ if n is odd.

The series of terms

$$_nC_0\,;\ _nC_1\,;\ _nC_2\,;\ _nC_3\,;\ \dots;\ _nC_r\,;\ \dots;\ _nC_{n-1}\,;\ _nC_n$$

can be written

$$1\,;\ \frac{n}{1}\,;\ \frac{n}{1}\cdot\frac{n-1}{2}\,;\ \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\,;\ \dots.$$

The multiplying factors which convert each term into the next, namely $\frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}, \dots$, steadily decrease; and the terms themselves steadily increase so long as the multiplying factors exceed 1, and steadily decrease as soon as the multiplying factors become less than 1.

But since $_nC_r={_nC_{n-r}}$, the terms which are equidistant from the beginning and end are equal. Therefore the greatest term is the middle one when n is even, and there are two equal greatest terms, namely the two middle terms, when n is odd : these are given respectively by $r=\frac{1}{2}n$ and by $r=\frac{1}{2}(n-1)$ or $\frac{1}{2}(n+1)$.

EXERCISE I. d

1. In how many ways can 6 different books be divided between A, B, C so that A has 3, B has 2 and C has 1 ?

2. Two elevens are made up from 22 players. In how many arrangements will 2 particular players be on opposite sides ?

3. In how many ways can 6 people bo divided into three pairs ?

4. How many selections can be made from 10 different books, if any number may be taken ?

5. How many whole numbers are factors of $2^4 . 3^4 . 5^2 . 7 . 11$, not counting 1 or the number itself ?

6. How many selections can be made from the letters, a, a, a, a, b, c, d, if any number may be taken ?

7 In how many ways can 144 be expressed as the product of 2 positive integers, counting 1×144 as one way ?

8. There are 6 points on a straight line and 8 other points. What is the greatest possible number of triangles that can ever be formed having these points as vertices ?

9. Prove that the number of ways of dividing up 20 unlike things into 5 packets of 4 each is to the number of ways of dividing them into 4 packets of 5 each as 125 is to 24.

10. 8 men are to play 4 singles at tennis, the games being simultaneous. In how many ways can the games be arranged ?

11. Two similar dice with faces numbered 1 to 6 are thrown. How many different (i) throws, (ii) totals are possible ?

12. In how many ways can a selection be made from 3 red balls, 4 blue balls, 5 green balls, if any number from 1 to 12 may be chosen ?

13. Two crews of 8 are formed from 16 oarsmen, of whom 4 can row only on bow side and 5 only on stroke side. In how many ways can this be done, regard being paid to the order in which they row ?

14. How many circles can be drawn so that each passes through 3 out of 9 given points (i) when no 4 of the points are concyclic, (ii) when 5 of the given points lie on one circle and the remaining 4 lie on another circle, no other set of 4 points being cyclic ?

15. 5 roads, A, B, C, D, E, meet at a junction, and a car comes along each road to this junction. In how many different ways may these cars continue their journeys across the junction ?

In how many of these cases do exactly 2 of the cars go down the road A ?

16. If ${}_{3n}C_r = {}_{3n}C_{n+r}$, express the value of each in terms of n.

17. Prove that ${}_nC_r + 2 \cdot {}_nC_{r-1} + {}_nC_{r-2} = {}_{n+2}C_r$.

18. Prove that the greatest value of ${}_{2n}C_r$ is double the greatest value of ${}_{2n-1}C_r$.

19. In how many ways can $2n$ people be divided into n couples?

20. Find the number of ways in which 6 men and 6 women can be arranged in 3 sets for tennis (i) if there is no restriction, (ii) if each man has a woman as partner.

21. There are $2n$ things of which n are alike and the rest are unlike. How many selections can be made, if any number may be taken ?

22. There are m points on one straight line AB and n points on another line AC, none of them being the point A. How many triangles can be formed with these points as vertices ? How many, if the point A is added?

MISCELLANEOUS EXAMPLES

EXERCISE I.

1. In how many ways can 9 different books be arranged on a shelf if two particular books are separated ?

2. In how many ways can n boys stand in a row if two particular boys are excluded from being at either end, and if two other boys must sit next to each other ?

3. In how many ways can three cards be selected from a pack of 52 cards, if at least one of them is to be an ace ?

4. Four letters are written and four envelopes addressed. In how many ways can all the letters be placed in the wrong envelopes ?

5. In how many ways can two sets of tennis, 4 players in each set, be arranged if 10 people are available ?

6. How many even numbers greater than 300 can be formed with the digits 1, 2, 3, 4, 5, no repetitions being allowed ?

7. Find the number of arrangements of letters in *alchemist* if the order of the consonants must not be changed.

8. Prove that there are about 40 million different arrangements of the letters in *inequalities*.

9. In how many ways can a committee of 2 Englishmen, 2 Frenchmen, 1 American be chosen from 6 Englishmen, 7 Frenchmen, 3 Americans ? In how many of these ways do a particular Englishman and a particular Frenchman belong to the committee ?

10. Attempts are made to predict the results (win, draw, lose) of 10 football matches. In how many different ways can exactly 6 correct results be given ?

11. Find the number of arrangements of the letters in *ratatatatat*. How many are there in which the a's are all separated from one another ?

12. A detachment consists of 3 sergeants, 4 corporals and 70 privates. In how many ways can a guard of 6 be selected to include 1 sergeant, 1 corporal and 4 privates ?

13. In how many ways can a lawn-tennis mixed double be made up from 5 married couples, if no husband and wife play in the same set ?

14. $2m$ white counters and $2n$ red counters are arranged in a straight line with $(m+n)$ counters on each side of a central mark. Find how many of the arrangements are symmetrical with respect to this mark.

II. BINOMIAL THEOREM

Product of Binomial Factors

The expansion of $(x+a_1)(x+a_2)$ is the sum of the terms obtained by multiplying together 2 letters, one taken from each of the 2 factors.

$$(x+a_1)(x+a_2)=x^2+xa_1+xa_2+a_1a_2.$$

It is convenient to group them according to the number of x factors in the terms.

$$(x+a_1)(x+a_2)=x^2+x(a_1+a_2)+a_1a_2.$$

Similarly,

$$\begin{aligned}&(x+a_1)(x+a_2)(x+a_3)\\ &=x^3+x^2(a_1+a_2+a_3)+x(a_2a_3+a_3a_1+a_1a_2)+a_1a_2a_3.\end{aligned}$$

Similarly,

$$\begin{aligned}&(x+a_1)(x+a_2)(x+a_3)(x+a_4)\\ &=x^4+x^3(a_1+a_2+a_3+a_4)\\ &\quad+x^2(a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4)\\ &\quad+x(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)+a_1a_2a_3a_4.\end{aligned}$$

This may be written

$$(x+a_1)(x+a_2)(x+a_3)(x+a_4)=x^4+s_1x^3+s_2x^2+s_3x+s_4,$$

where $s_1=$ the sum of the terms a_1, a_2, a_3, a_4,

$s_2=$ the sum of their products, two at a time,

$s_3=$ the sum of their products, three at a time,

$s_4=$ the product, four at a time, $a_1a_2a_3a_4$.

This process can be applied to as many factors as required.

Thus $(x+a_1)(x+a_2)(x+a_3)\ldots(x+a_n)$

$$=x^n+s_1x^{n-1}+s_2x^{n-2}+s_3x^{n-3}+\ldots+s_{n-1}x+s^n.$$

where s_1 contains n terms, $(a_1+a_2+\ldots+a_n)$,

s_2 contains ${}_nC_2$ terms, $(a_1a_2+a_1a_3+a_2a_3+\ldots)$,

because it consists of the selections, two at a time, from n unlike things,

s_3 contains ${}_nC_3$ terms, $(a_1a_2a_3+a_1a_2a_4+\dots)$,

and, in general, s_r contains ${}_nC_r$ terms.

If $a_1=a_2=a_3=\dots=a_n=a$, the left side becomes $(x+a)^n$.

Also, $s_1=a+a+\dots$, n terms, $=na$;

$s_2=a^2+a^2+\dots$, ${}_nC_2$ terms, $={}_nC_2a^2$;

$s_3=a^3+a^3+\dots$, ${}_nC_3$ terms, $={}_nC_3a^3$;

and so on.

Therefore, if n is any positive integer,

$$(x+a)^n=x^n+nx^{n-1}a+{}_nC_2x^{n-2}a^2+\dots+{}_nC_rx^{n-r}a^r+\dots+a^n$$

$$=x^n+nx^{n-1}a+\frac{n(n-1)}{1.2}x^{n-2}a^2+\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3+\dots$$

$$+\frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r}x^{n-r}a^r+\dots+a^n.$$

This is called the Binomial Theorem for a positive integral index.

It may also be proved by "induction." The method of induction consists in showing that, *if* the theorem is true for some special integral value of n, say $n=k$, then it is true for $n=k+1$. Consequently, if it is true for $n=2$, then it is true for $n=3$, and therefore for $n=4$, and so on indefinitely.

Suppose that the theorem is true for $n=k$, that is, suppose that

$$(x+a)^k=x^k+{}_kC_1x^{k-1}a+{}_kC_2x^{k-2}a^2+\dots$$
$$+{}_kC_{r-1}x^{k-r+1}a^{r-1}+{}_kC_rx^{k-r}a^r+\dots+a^k.$$

Then $(x+a)^{k+1}=(x+a)(x+a)^k$

$$=x^{k+1}+(1+{}_kC_1)x^ka+({}_kC_1+{}_kC_2)x^{k-1}a^2+\dots$$
$$+({}_kC_{r-1}+{}_kC_r)x^{k-r+1}a^r+\dots+a^{k+1}.$$

But ${}_kC_{r-1}+{}_kC_r={}_{k+1}C_r$, see p. 13A,

$$\therefore\ (x+a)^{k+1}=x^{k+1}+{}_{k+1}C_1x^ka+{}_{k+1}C_2x^{k-1}a^2+\dots$$
$$+{}_{k+1}C_rx^{k-r+1}a^r+\dots+a^{k+1}.$$

$\therefore$ *if* the theorem is true for $n=k$, it is true for $n=k+1$.

But it is true when $n=2$ because

$$(x+a)^2=x^2+2xa+a^2=x^2+{}_2C_1xa+a^2,$$

$\therefore$ it is true when $n=3$, and $\therefore$ when $n=4$, and so on indefinitely.

$\therefore$ it is true for all positive integral values of n.

Properties of the Binomial Expansion

(i) The expansion of $(x+a)^n$ contains $(n+1)$ terms, of which ${}_nC_r x^{n-r}a^r$ is the $(r+1)$th term; this term or the rth term is called the *general term*.

Also the coefficients of terms equidistant from the two ends are equal, for the $(r+1)$th term from the end is ${}_nC_{n-r}x^r a^{n-r}$, and ${}_nC_r = {}_nC_{n-r}$, see p. 12A.

(ii) If n is even, there is one middle term given by putting $r=\frac{1}{2}n$ in ${}_nC_r x^{n-r}a^r$. If n is odd, there are two middle terms given by putting $r=\frac{1}{2}(n-1)$ and $r=\frac{1}{2}(n+1)$.

(iii) If $a=1$, we have

$$(1+x)^n = 1 + {}_nC_1 x + {}_nC_2 x^2 + \dots + {}_nC_r x^r + \dots + x^n.$$

(iv) If we write $-a$ for a, we have

$$(x-a)^n = x^n - {}_nC_1 x^{n-1}a + {}_nC_2 x^{n-2}a^2 - \dots$$
$$+ (-1)^r {}_nC_r x^{n-r}a^r + \dots + (-1)^n a^n.$$

Some practice in writing down binomial expansions has been given in Chapter VIII., p. 123. For convenience, further examples are added here.

EXERCISE II. a

[*Large coefficients should be left in factors, not multiplied out.*]

1. Write down the expansions of

   (i) $(x+a)(x+b)(x+c)$; (ii) $(x+2a)(x+2b)(x+2c)$;

   (iii) $(x+a)(x+b)(x+c)(x+d)$.

2. What is the coefficient of x^3 in the expansion of

$$(x+a_1)(x+a_2)(x+a_3)(x+a_4)(x+a_5)\ ?$$

How many terms are there in the coefficient of x^2 ? What are the coefficients of x^3 and x^2 in $(x+a)^5$?

3. How many terms are there in the coefficients of x^{n-2}, x^2, x^r in the expansion of

$$(x+a_1)(x+a_2)(x+a_3)\dots(x+a_n)\ ?$$

What are the coefficients of x^{n-2}, x^2, x^r in $(x+a)^n$?

Write down the expansions of Nos. 4-11 :

4. $(x+1)^4$. 5. $(y-1)^4$. 6. $(x+2a)^3$. 7. $(z+2)^4$.

8. $(a-b)^5$. 9. $\left(y+\frac{1}{y}\right)^6$. 10. $(3x-2y)^4$. 11. $(x^2+3y^2)^5$.

12. For the expansion of $(x+1)^8$, find (i) the number of terms, (ii) which term involves x^4, (iii) the power of x in the 7th term, (iv) the coefficient of x^5.

13. For the expansion of $(5x-3y)^{20}$, find (i) the number of terms, (ii) the powers of x and y in the 6th term and in the 12th term, (iii) the 3rd term, the 19th term, the middle term.

14. Find the coefficient of x^2 in the expansions of

(i) $(1+2x)^9$; (ii) $(1-3x)^7$; (iii) $(x+3)^6$; (iv) $(2x-5)^8$.

15. Find the coefficient of x^3 in the expansions of

(i) $(x-10)^5$; (ii) $(2x-\frac{1}{2})^6$; (iii) $(x-y)^{10}$; (iv) $\left(x+\dfrac{1}{x}\right)^7$.

16. Find the 4th term in the expansions of

(i) $(2a-5b)^7$; (ii) $(1+3x)^n$; (iii) $\left(x+\dfrac{1}{x}\right)^n$.

17. Find the $(r+1)$th term in the expansions of

(i) $(x+y)^n$; (ii) $(a-2b)^n$; (iii) $\left(x+\dfrac{1}{x}\right)^n$.

18. Expand in ascending powers of x:

(i) $(1+x)^n+(1-x)^n$; (ii) $(1+x)^n-(1-x)^n$.

19. Evaluate $(1{\cdot}0025)^{10}$ correct to 6 decimal places.

20. Use the binomial expansion to show that £1000 will amount to £1480, to the nearest £, in 10 years at 4 per cent. per annum compound interest.

21. What is the coefficient of x^{20} in $(x^2+3x)^{12}$?

22. What is the coefficient of x^n in $(1+\frac{1}{4}x)^{2n}$?

23. (i) What is the middle term of the expansion of $\left(x+\dfrac{1}{x}\right)^{2n}$?

(ii) What are the two middle terms for $\left(x-\dfrac{1}{x}\right)^{2n+1}$?

24. Simplify $(a+b)^3-3b(a+b)^2+3b^2(a+b)-b^3$.

25. Simplify $(x-1)^4+4(x-1)^3+6(x-1)^2+4(x-1)+1$.

26. Find the coefficient of x^4 in the expansions of

(i) $(1-x)(1+x)^5$; (ii) $(1+x)(1-x)^n$.

27. Find the first 4 terms in the expansion in ascending powers of x of $(1+2x)(1-x^3)^9$. What is the coefficient of x^{19}?

28. Simplify $(\sqrt{2}+1)^5-(\sqrt{2}-1)^5$.

29. Find (i) the sum, (ii) the product of $(2+\sqrt{3})^7$ and $(2-\sqrt{3})^7$. Hence show that the integral part of $(2+\sqrt{3})^7$ is 10,083.

30. Simplify $x^n(x-1)^n+{}_nC_1x^{n-1}(x-1)^{n-1}(x+1)$

$+\dots+{}_nC_rx^{n-r}(x-1)^{n-r}(x+1)^r+\dots(x+1)^n$.

Special Terms

The following examples illustrate the selection of particular terms from an expansion.

Example 1. Find the term independent of x in the expansion of $\left(3x - \frac{5}{x^3}\right)^8$.

The general term is ${}_8C_r(3x)^{8-r}\left(-\frac{5}{x^3}\right)^r$ in which the power of x is $8 - 4r$. Taking $r = 2$, we get ${}_8C_2(3x)^6\left(-\frac{5}{x^3}\right)^2$, which equals

$$\frac{8 . 7}{1 . 2} . 3^6 . 5^2 = 700 \times 3^6.$$

Alternatively, as follows:

$$\left(3x - \frac{5}{x^3}\right)^8 = \left(\frac{3x^4 - 5}{x^3}\right)^8 = \frac{(3x^4 - 5)^8}{x^{24}}.$$

$\therefore$ we require the coefficient of x^{24} in $(3x^4 - 5)^8$.

The term in x^{24} is ${}_8C_2(3x^4)^6(-5)^2$;

$$\therefore \text{ the coefficient is } \frac{8 . 7}{1 . 2} . 3^6 . 5^2 = 700 \times 3^6.$$

Example 2. Expand $(2 + 3x - x^2)^n$ in ascending powers of x as far as x^2.

$$\begin{aligned}(2 + 3x - x^2)^n &= [2 + x(3 - x)]^n \\ &= 2^n + n . 2^{n-1}x(3 - x) + \frac{n(n-1)}{1 . 2} . 2^{n-2}x^2(3 - x)^2 + \dots \\ &= 2^n + 3nx . 2^{n-1} - nx^2 . 2^{n-1} + (n^2 - n) . 2^{n-3} . 9x^2 + \dots \\ &= 2^n + 3nx . 2^{n-1} + (9n^2 - 13n)\, x^2 . 2^{n-3} + \dots .\end{aligned}$$

Example 3. Find which is the numerically greatest term in the expansion of $(5 - 4x)^{12}$ when $x = \frac{2}{3}$.

The terms in the expansion are alternately positive and negative, but the *numerically* greatest term will be the same term as in the expansion of $(5 + 4x)^{12}$.

If u_r denotes the rth term, $u_{r+1} > u_r$ if $\frac{12 - r + 1}{r} . \frac{4x}{5} > 1$ or if $8(13 - r) > 15r$ since $x = \frac{2}{3}$.

$$\therefore u_{r+1} > u_r \text{ if } 104 - 8r > 15r \text{ or } 23r < 104 \text{ or } r < 4\tfrac{12}{23}.$$

$$\therefore u_5 > u_4 > u_3 \dots \text{ and } u_5 > u_6 > u_7 \dots .$$

$$\therefore \text{ the 5th term is the greatest.}$$

EXERCISE II. b

[*Large coefficients should be left in factors, not multiplied out.*]

1. Write down the named terms in the following expansions :

(i) 4th term in $(2x-3y)^7$; (ii) 10th term in $\left(3x+\frac{2}{x}\right)^{12}$;

(iii) 6th term in $\left(\frac{x^2}{2}-\frac{x}{3}\right)^8$; (iv) rth term in $\left(x+\frac{1}{x}\right)^n$.

2. Write down the coefficients of tho named terms in the following expansions :

(i) x^8 in $(2-x)^{11}$; (ii) x^9 in $\left(2x-\frac{3}{x}\right)^{13}$;

(iii) x^{11} in $(3x+2x^2)^9$; (iv) x^{2r} in $(1-x^2)^{4r}$;

(v) x^{r+1} in $(1-x)^{n-1}$; (v) x^{2r} in $\left(x+\frac{1}{x}\right)^{4r}$.

3. Write down the terms independent of x in the following expansions :

(i) $\left(x-\frac{2}{x}\right)^{10}$; (ii) $\left(2x^3-\frac{1}{x}\right)^{12}$; (iii) $\left(x+\frac{1}{x}\right)^{2n}$.

4 Write down the coefficients of the named terms in the following expansions :

(i) x^2 in $\left(2x^2-\frac{1}{2x}\right)^{13}$; (ii) x^3 in $\left(x^4-\frac{1}{2x}\right)^{12}$;

(iii) x^7 in $\left(2x-\frac{3}{x}\right)^{15}$; (iv) x^7 in $\left(2x^2-\frac{1}{4x}\right)^{11}$.

5. Expand $(1-2x+3x^2)^7$ in ascending powers of x as far as x^3.

6. Find the coefficient of x^4 in the expansion of $(2+x-x^2)^4$.

7. Find the coefficient of x^7 in the expansion of $(1-x^2-x^3)^n$.

8. What is the ratio of the $(r+1)$th term to the rth term in the expansions of

(i) $\left(1+\frac{x}{2}\right)^n$; (ii) $\left(x-\frac{3}{x}\right)^n$; (iii) $(2x+3y)^{2r}$?

9. What is the ratio of the $(r+1)$th term to the $(r-1)$th term in the expansions of (i) $(x-y)^n$; (ii) $(2x+3y)^n$?

10. Find the value of r if the coefficients of x^r and of x^{r+1} in $(3x+2)^{19}$ are equal.

11. If $x=0{\cdot}2$, prove that the 11th term in the expansion of $(1+x)^{14}$ is $\frac{1}{10}$th of the 10th term.

12. What is the greatest coefficient in the expansions of

(i) $(1+x)^{10}$; (ii) $(1+x)^{11}$; (iii) $(1+x)^{4n+3}$?

13. In the expansion of $(1+x)^{12}$, the ratio of a certain coefficient to the preceding coefficient is $\frac{6}{7}$; find the numerical values of the two coefficients.

14. Find which term in the expansion of $\left(x+\dfrac{1}{2x}\right)^{3n}$ has the largest coefficient.

15. Find which are the numerically greatest terms in the expansions of the following :

(i) $(1+2x)^9$ if $x=\frac{1}{3}$; (ii) $(1+2x)^9$ if $x=3$;
(iii) $(7+x)^{29}$ if $x=3$; (iv) $(1+3x)^7$ if $x=\frac{1}{4}$;
(v) $(ax-by)^{10}$ if $a=2$, $b=5$, $x=3$, $y=\frac{1}{2}$?

16. The expression $(3+5)^{20}$ is expanded by the binomial theorem. Prove that each of the 2nd, 3rd, ... , 10th terms is more than double the term which precedes it, and that each of the 11th, 12th, ... , 21st terms is less than double the term which precedes it.

17. Prove that the coefficients of x^2 and x^3 in the expansion of $(x^2+2x+2)^n$ are $2^{n-1}.n^2$ and $\frac{1}{3}n(n^2-1).2^{n-1}$.

18. Find the term independent of x in the expansion of

$$\left(x+\frac{1}{x}\right)^3\left(x-\frac{1}{x}\right)^5.$$

19. Expand $\left(x+\dfrac{1}{x}\right)^4\left(x-\dfrac{1}{x}\right)^3$ in descending powers of x.

20. Expand $(1+x+x^2)^n$ in ascending powers of x as far as x^3.

21. If a_r is the coefficient of x^r in $(1+bx^2+cx^3)^n$, prove that $2na_4=(n-1)a_2^2$.

22. Find the coefficient of x^r in the expansion of

(i) $(1-x)(1+x)^n$; (ii) $(1+2x+x^2)(1+x)^n$.

Relations between Binomial Coefficients

For the rest of the chapter, we shall for simplicity denote ${}_nC_r$ by c_r.

Thus $(1+x)^n \equiv c_0+c_1x+c_2x^2+\dots+c_rx^r+\dots+c_nx^n$.

Here, $c_0, c_1, c_2, \dots, c_n$ are called *binomial coefficients.*

In the expansion of $(1+x)^n$,

(i) the sum of the coefficients is 2^n,

(ii) the sum of the coefficients of the odd terms equals that of the even terms, each being 2^{n-1}.

(i) In the expansion of $(1+x)^n$, put $x=1$;

$$\therefore (1+1)^n = c_0+c_1+c_2+\dots+c_n ;$$

$$\therefore c_0+c_1+c_2+\dots+c_n = 2^n.$$

(ii) In the expansion of $(1+x)^n$, put $x=-1$;

$$\therefore\ (1-1)^n = c_0 - c_1 + c_2 - \dots + (-1)^n c_n;$$

$$\therefore\ c_0 + c_2 + c_4 + \dots = c_1 + c_3 + c_5 + \dots .$$

But the sum of all the coefficients is 2^n,

$$\therefore\ c_0 + c_2 + c_4 + \dots = (\tfrac{1}{2} \text{ of } 2^n) = 2^{n-1}.$$

Note. An alternative proof of (i) was given on p. 16A.

Finite Series involving Binomial Coefficients

The most useful methods of summation may be classified as follows:

(i) Express the series as the sum of two or more binomial expansions.

(ii) Obtain the series by differentiation or integration of a finite series whose sum is known.

(iii) Build up a function in which the given series is the coefficient of a particular power of the variable and evaluate this coefficient in an independent manner.

The following examples illustrate these methods.

Example 4. Sum the series

$$c_0 + 2c_1x + 3c_2x^2 + \dots + (n+1)c_nx^n.$$

First Method

The series $= (c_0 + c_1x + c_2x^2 + \dots + c_nx^n) + (c_1x + 2c_2x^2 + \dots + nc_nx^n)$.

The first bracket $= (1+x)^n$;

the second bracket

$$= nx + 2\cdot\frac{n(n-1)}{2!}x^2 + 3\cdot\frac{n(n-1)(n-2)}{3!}x^3 + \dots + nx^n$$

$$= nx\left\{1 + (n-1)x + \frac{(n-1)(n-2)}{2!}x^2 + \dots + x^{n-1}\right\}$$

$$= nx(1+x)^{n-1}.$$

$$\therefore\ \text{the series} = (1+x)^n + nx(1+x)^{n-1}.$$

Second Method

$$c_0x + c_1x^2 + c_2x^3 + \dots + c_nx^{n+1} \equiv x(1+x)^n.$$

Differentiate each side w.r.t. x;

$$\therefore\ c_0 + 2c_1x + 3c_2x^2 + \dots + (n+1)c_nx^n = (1+x)^n + nx(1+x)^{n-1}.$$

Note. The sum of the series

$$c_0+2c_1+3c_2+\ldots+(n+1)c_n$$

is obtained by putting $x=1$.

The sum $=(1+1)^n+n(1+1)^{n-1}=2^n+n\,.\,2^{n-1}=(n+2)\,.\,2^{n-1}$.

Example 5. Find the value of

(i) $c_0^2+c_1^2+c_2^2+\ldots+c_n^2$; (ii) $c_0c_1+c_1c_2+c_2c_3+\ldots+c_{n-1}c_n$.

$$(c_0+c_1x+c_2x^2+\ldots+c_nx^n)(c_0+c_1x+c_2x^2+\ldots+c_nx^n)\equiv(1+x)^n(1+x)^n.$$

(i) The coefficient of x^n on the left side

$$\begin{aligned}&=c_0c_n+c_1c_{n-1}+c_2c_{n-2}+\ldots+c_nc_0\\&=c_0^2+c_1^2+c_2^2+\ldots+c_n^2,\text{ since } c_r=c_{n-r}.\end{aligned}$$

$\therefore$ $c_0^2+c_1^2+\ldots+c_n^2$ equals the coefficient of x^n in $(1+x)^{2n}$, and this is ${}_{2n}C_n=\dfrac{(2n)!}{n!\,n!}$.

$$\therefore\ c_0^2+c_1^2+\ldots+c_n^2=\frac{(2n)!}{n!\,n!}.$$

(ii) The coefficient of x^{n-1} on the left side

$$\begin{aligned}&=c_0c_{n-1}+c_1c_{n-2}+c_2c_{n-3}+\ldots+c_{n-1}c_0\\&=c_0c_1+c_1c_2+c_2c_3+\ldots+c_{n-1}c_n,\text{ since } c_r=c_{n-r}\\&=\text{coefficient of } x^{n-1}\text{ in }(1+x)^{2n}\\&={}_{2n}C_{n-1}=\frac{(2n)!}{(n-1)!\,(n+1)!}.\end{aligned}$$

Example 6. Sum the series

$$c_1+2^2c_2x+3^2c_3x^2+4^2c_4x^3+\ldots+n^2c_nx^{n-1}$$

and deduce the value, if $n>2$, of

$$c_1-2^2c_2+3^2c_3-\ldots+(-1)^{n-1}n^2c_n.$$

$$c_0+c_1x+c_2x^2+c_3x^3+\ldots+c_nx^n\equiv(1+x)^n.$$

Differentiate w.r.t. x,

$$\therefore\ c_1+2c_2x+3c_3x^2+\ldots+nc_nx^{n-1}\equiv n(1+x)^{n-1};$$

$$\therefore\ c_1x+2c_2x^2+3c_3x^3+\ldots+nc_nx^n\equiv nx(1+x)^{n-1}.$$

Differentiate w.r.t. x,

$$\begin{aligned}\therefore\ c_1+2^2c_2x+3^2c_3x^2+\ldots+n^2c_nx^{n-1}&\\ \equiv n(1+x)^{n-1}+n(n-1)x(1+x)^{n-2};&\end{aligned}$$

$$\therefore\ \text{the sum}=n(1+x)^{n-2}(1+nx).$$

If $x=-1$, $(1+x)^{n-2}=0$ for $n>2$;

$$\therefore\ c_1-2^2c_2+3^2c_3-\ldots+(-1)^{n-1}n^2c_n=0.$$

EXERCISE II. c

1. What relations are obtained by putting (i) $x=2$, (ii) $x=-2$ in the expansion of $(1+x)^n$?

2. What is the sum of the coefficients of the powers of x, including x^0, in the expansion of (i) $(1+x)^8$; (ii) $(1+x+x^2)^4$?

3. If $(1+2x+2x^2)^3\equiv a_0+a_1x+a_2x^2+\ldots+a_6x^6$, find the values of (i) $a_0+a_1+a_2+\ldots+a_6$; (ii) $a_0+a_2+a_4+a_6$; (iii) $a_1+a_3+a_5$.

4. Expand $(1-2x+3x^2)^5$ in ascending powers of x as far as x^4. Find in the complete expansion (i) the algebraic sum of the coefficients of all the powers of x, (ii) the sum of all these coefficients, each taken positively, (iii) the algebraic sum of the coefficients of all the odd powers of x.

5. Prove that $(1+c_1+c_2+\ldots+c_n)^2$ equals

$$1+{}_{2n}C_1+{}_{2n}C_2+\ldots+{}_{2n}C_{2n}.$$

6. Obtain relations by equating the coefficients of x^r in the expansions of each side of the following identities:

(i) $(1+x)^{n+1}\equiv(1+x)(c_0+c_1x+\ldots+c_nx^n)$;

(ii) $(1+x)^{n+2}\equiv(1+2x+x^2)(c_0+c_1x+\ldots+c_nx^n)$;

(iii) $(1+x)^{n+3}\equiv(1+3x+3x^2+x^3)(1+x)^n$.

7. Prove that (i) $c_1+2c_2+3c_3+\ldots+nc_n=n\,.\,2^{n-1}$.

(ii) $c_1-2c_2+3c_3+\ldots+(-1)^{n-1}nc_n=0$.

8. What is the value of $c_0-2c_1+3c_2-\ldots+(-1)^n(n+1)c_n$?

9. Sum the series $c_0x+c_1\dfrac{x^2}{2}+c_2\dfrac{x^3}{3}+\ldots+c_n\dfrac{x^{n+1}}{n+1}$.

Prove that $c_0+\frac{1}{2}c_1+\frac{1}{3}c_2+\ldots+\dfrac{1}{n+1}c_n=\dfrac{1}{n+1}(2^{n+1}-1)$.

[For a calculus method, use integration.]

10. Sum the series

(i) $c_0+c_2x^2+c_4x^4+\ldots+c_nx^n$, n even;

(ii) $c_0+c_2x^2+c_4x^4+\ldots+c_{n-1}x^{n-1}$, n odd.

11. Use the identity $(1+x)^n(1-x)^n\equiv(1-x^2)^n$ to show that

$$c_0^2-c_1^2+c_2^2-\ldots+(-1)^nc_n^2=(-1)^{\frac{1}{2}n}\frac{n!}{(\frac{1}{2}n)!\,(\frac{1}{2}n)!}$$

if n is even.

What is its value if n is odd?

12. Prove that $c_0c_r - c_1c_{r-1} + c_2c_{r-2} - \dots + (-1)^r c_r c_0$ equals

$$(-1)^{\frac{1}{2}r}\frac{n!}{(\frac{1}{2}r)!\,(n-\frac{1}{2}r)!} \text{ if } r \text{ is even.}$$

What is its value if r is odd?

13. Use the identity $(1+x)^m(1+x)^n = (1+x)^{m+n}$ to prove *Vandermonde's* theorem,

$${}_m\mathrm{C}_r + {}_m\mathrm{C}_{r-1}\,{}_n\mathrm{C}_1 + {}_m\mathrm{C}_{r-2}\,{}_n\mathrm{C}_2 + \dots + {}_n\mathrm{C}_r = {}_{m+n}\mathrm{C}_r.$$

14. Prove that

$$c_1 + 3c_3 + 5c_5 + \dots = 2c_2 + 4c_4 + 6c_6 + \dots = n \,.\, 2^{n-2}.$$

15. Prove that $c_0 + 3c_1 + 5c_2 + \dots + (2n+1)c_n = (n+1)\,.\,2^n$.

16. Prove that $c_2 + 2c_3 + 3c_4 + \dots + (n-1)c_n = 1 + (n-2)\,.\,2^{n-1}$.

17. Prove that $c_1^2 + 2c_2^2 + 3c_3^2 + \dots + nc_n^2 = \dfrac{(2n-1)!}{(n-1)!\,(n-1)!}$.

18. Prove that

$$c_0^2 + 2c_1^2 + 3c_2^2 + \dots + (n+1)c_n^2 = \frac{(n+2)\,.\,(2n-1)!}{n!\,(n-1)!}.$$

19. Find the sum of the coefficients in the expansion of

(i) $(1+2x)^5$; (ii) $(3x+7y)^6$; (iii) $(1+x+x^2)^4$.

20. Prove that

$$a + c_1(a+d)x + c_2(a+2d)x^2 + \dots + c_n(a+nd)x^n$$

equals $$(1+x)^{n-1}[a(1+x)+ndx].$$

21. Expand $x^p(1+x^q)^n$, where p, q, n are positive integers, in powers of x. What identity is obtained by differentiating w.r.t. x? Deduce special results by putting (i) $x=1$, (ii) $x=-1$.

If m is a positive integer, the series

$$1 + mx + \frac{m(m-1)}{1\,.\,2}x^2 + \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}x^3 + \dots$$

terminates automatically after $(m+1)$ terms, since each later coefficient contains the factor $(m-m)=0$; and the sum of this *finite* series is $(1+x)^m$, (see Ch. II, p. 22A).

If m is a **positive fraction or any negative number**, the series does not terminate. For example, the series becomes,

if $m=-1$, $1 - x + x^2 - x^3 + x^4 - \dots$;

if $m=-2$, $1 - 2x + 3x^2 - 4x^3 + \dots$;

if $m=-\frac{1}{2}$, $1 - \frac{1}{2}x + \dfrac{1\,.\,3}{2\,.\,4}x^2 - \dfrac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^3 + \dots$;

if $m=\frac{1}{2}$, $1 + \frac{1}{2}x + \dfrac{(-1)\,.\,1}{2\,.\,4}x^2 - \dfrac{(-1)\,.\,1\,.\,3}{2\,.\,4\,.\,6}x^3 + \dfrac{(-1)\,.\,1\,.\,3\,.\,5}{2\,.\,4\,.\,6\,.\,8}x^4 - \dots$

Series of this form are called **binomial series.**

The Binomial Theorem for a Fractional or Negative Index

If m is a positive fraction or any negative number, and if $|x| < 1$, the sum to infinity of

$$1+mx+\frac{m(m-1)}{1.2}x^2+\frac{m(m-1)(m-2)}{1.2.3}x^3+\dots$$

is the POSITIVE value of $(1+x)^m$.

Note. The notation $|x| < 1$ is used to mean that $-1 < x < 1$; similarly $|x| < \frac{1}{2}$ means $-\frac{1}{2} < x < \frac{1}{2}$; and so on.

The formal proof of the Binomial Theorem is given in *Advanced Algebra*, Vol. I, Ch. V; but applications of it can be practised without difficulty.

Example 7. Expand $(1-x)^{-3}$ given $|x| < 1$.

$$(1-x)^{-3}=1+(-3)(-x)+\frac{(-3)(-4)}{1.2}(-x)^2$$
$$+\frac{(-3)(-4)(-5)}{1.2.3}(-x)^3+\dots$$
$$=1+3x+\frac{3.4}{1.2}x^2+\frac{4.5}{1.2}x^3+\dots.$$

Example 8. Expand $(1+2x)^{\frac{5}{3}}$ given $|x|<\frac{1}{2}$.

The function can be expanded in ascending powers of x if $|2x|<1$, that is, if $|x|<\frac{1}{2}$. The first 4 terms are

$$1+\tfrac{5}{3}(2x)+\frac{\frac{5}{3}\cdot\frac{2}{3}}{1.2}(2x)^2+\frac{\frac{5}{3}\cdot\frac{2}{3}\cdot(-\frac{1}{3})}{1.2.3}(2x)^3$$

$$\text{or}\quad 1+\frac{5}{1}\left(\frac{2x}{3}\right)+\frac{5.2}{1.2}\left(\frac{2x}{3}\right)^2+\frac{5.2.(-1)}{1.2.3}\left(\frac{2x}{3}\right)^3.$$

The terms which follow are alternately positive and negative, and the general term is

$$(-1)^r\frac{5.2.1.4.7\dots(3r-8)}{1.2.3.4.5\dots r}\left(\frac{2x}{3}\right)^r,\quad r>2.$$

Example 9. Expand $\dfrac{1}{\sqrt{(1+x^2)}}$ given $x > 1$.

$$\frac{1}{\sqrt{(1+x^2)}} = \frac{1}{x}\left(1+\frac{1}{x^2}\right)^{-\frac{1}{2}}; \text{ but } 0 < \frac{1}{x} < 1,$$

$$\therefore \frac{1}{\sqrt{(1+x^2)}} = \frac{1}{x}\left\{1 - \frac{1}{2}\cdot\frac{1}{x^2} + \frac{1.3}{2.4}\cdot\frac{1}{x^4} - \frac{1.3.5}{2.4.6}\cdot\frac{1}{x^6} + \ldots\right\}$$

$$= \frac{1}{x} - \frac{1}{2}\cdot\frac{1}{x^3} + \frac{1.3}{2.4}\cdot\frac{1}{x^5} - \frac{1.3.5}{2.4.6}\cdot\frac{1}{x^7} + \ldots .$$

EXERCISE II. d

Write down the first 4 terms and the coefficient of x^r in the expansions in ascending powers of x of the following functions. In each case, state when the expansion is valid.

1. $(1-x)^{-2}$.
2. $(1+x)^{-3}$.
3. $(1-3x)^{-\frac{1}{2}}$.
4. $(1-5x)^{-1}$.
5. $(1-2x)^{-4}$.
6. $(1+4x)^{\frac{1}{2}}$.
7. $(1-x^2)^{-\frac{3}{2}}$.
8. $(1+3x)^{\frac{2}{3}}$.
9. $\sqrt{(4+x^2)}$.
10. $(1-x)^{-n}$.
11. $\dfrac{x}{1+x^2}$.
12. $(a+x)^{-2}$.
13. $\dfrac{x}{\sqrt{(a^2-x^2)}}$.
14. $\sqrt[3]{(8-x^3)^2}$.
15. $(1-nx)^{-\frac{1}{n}}$.

Find the named terms in the expansions of the following:

16. 10th term in $(1-4x)^{-3}$ if $|x| < \frac{1}{4}$.
17. 9th term in $(1-5x)^{-2}$ if $|x| > \frac{1}{5}$.
18. 4th term in $(1+3x)^{2\frac{1}{2}}$ if $|x| < \frac{1}{3}$.
19. 5th term in $(1+4x^2)^{-2\frac{1}{2}}$ if $|x| > \frac{1}{2}$.
20. 6th term in $(1-3x^2)^{-n}$ if $|x| < \frac{1}{2}$.